A constructive theory of ordinals

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Abstract

Martin-Löf [1970] describes "recursively constructed ordinals". He gives a constructively acceptable version of Kleene's computable ordinals. In fact, the Turing definition of computable functions is not needed from a constructive point of view.
We give in this paper a constructive theory of ordinals that is similar to Martin-Löf's theory but more detailed. In our setting, the operation "upper bound of two ordinals" plays an important rôle through its interactions with the two relations "$x \leqslant y$" and "$x < y$". This allows us to approach as much as we may the notion of *linear order* when the property "$\alpha \leqslant \beta$ or $\beta \leqslant \alpha$" is provable only within classical logic. Our problem is to give a formal definition corresponding to intuition, and to prove that our "constructive ordinals" satisfy constructively all desirable properties.

Contents

1 Introduction

In classical mathematics, a natural definition for an ordinal is to be an order type of a well-ordered set (cf. Bourbaki, *Theory of sets*). Nevertheless it is more convenient to use von Neumann ordinals, for which many results can be proved without using choice (see, e.g., Krivine [1998]).

Let us now propose a constructive approach. A binary relation $<$ on a set X is said to be *well-founded* when for any family of sets $(E_x)_{x\in X}$ indexed by X, it is possible to construct elements of $\sum_{x\in X} E_x$ by $<$-induction. Precisely, each time a construction γ is given, which from an element $a \in X$ and an element $\varphi \in \sum_{x\in X, x<a} E_x$ constructs an element $\gamma(a,\varphi) \in E_a$, there exists a unique $\phi \in \sum_{x\in X} E_x$ such that for all $a \in X$ we have $\phi(a) = \gamma(a, \phi|_{x\in X, x<a})$.

This notion has a clear constructive meaning.

In particular, let us consider a property for elements in X. If the property is $<$-hereditary, i.e., if it is true for $x \in X$ as soon as it is true for all $y \in X$ with $y < x$, then this property is true for all elements in X.

In constructive mathematics, Mines, Richman, and Ruitenburg [1988] define an ordinal as a linearly ordered set for which the order relation is well-founded. So all subsets of $\mathbb{N}$ are ordinals, even if we don't know whether they have a smallest element.

Among other constructive points of view there are descriptions of "countable ordinals constructed by induction" given by Brouwer [1926], by Gentzen [1936], by Heyting [1961], and by Martin-Löf [1970].

Brouwer proposes an inductive construction based on the idea that when "ordinals" $\alpha_n \geqslant 1$ are defined for all $n \in \mathbb{N}$ and are linearly ordered well-founded countable sets, then we can describe the ordinal α corresponding intuitively to α_1 "followed by" α_2 "followed by" α_3 "followed by" .... The ordered set α defined by Brouwer will again be a linearly ordered well-founded set. And if the order relation on each α_i is decidable, the same is true for α.

Two Brouwer ordinals are in general not comparable (within intuitionistic logic): there is no general criterion allowing us to decide whether two ordinals have the same order type, and, when this is not the case, which is isomorphic to an initial segment of the other.

Martin-Löf describes "recursively constructed ordinals". He gives a constructively acceptable version of Kleene's computable ordinals. Intuitively, an ordinal à la Martin-Löf is inductively defined using two basic constructions:

- there is a minimum ordinal $\underline{0}$;
- if $(\alpha_n)_{n\in\mathbb{N}}$ is an explicit sequence of ordinals (indexed by $\mathbb{N}$ or by an $\mathbb{N}_k = \{\, n \in \mathbb{N} \mid n < k \,\}$), the upper bound of the $\alpha_n + 1$'s is an ordinal[1].

The fact that the definition is inductive means that any ordinal is constructed using the indicated rules.

[1] Martin-Löf notes this upper bound $\sup_{n\in\mathbb{N}}(\alpha_n)$. This allows him to consider the empty sequence, with upper bound $\underline{0}$. Except for this case, it is the upper bound of the $\alpha_n + 1$'s. Thus we prefer the notation $\mathrm{succ}_{n\in\mathbb{N}}(\alpha_n)$.

Naturally we can drop Turing machines and replace Turing computability by intuitive (undefined) computability. In this case, the main difference between Brouwer and Martin-Löf ordinals is that Martin-Löf ordinals, being defined in a "parallel" way rather than in a "sequential" way, are more general: it is possible to construct the upper bound of ordinals $\alpha_n + 1$ of any sequence of well-defined ordinals. A drawback is that there is no way to associate to a Martin-Löf ordinal a linearly ordered well-founded set with the same order type. E.g., if the α_n are all equal to 0 or 1, it is a priori impossible to decide if α equals 1 or 2.

We give in this paper a constructive theory of ordinals that is similar to Martin-Löf's theory but more detailed. In our setting, the operation "upper bound of two ordinals" plays an important rôle through its interactions with the two relations "$x \leqslant y$" and "$x < y$". This allows us to approach as much as we may the notion of *linear order* when the property "$\alpha \leqslant \beta$ or $\beta \leqslant \alpha$" is provable only within classical logic. Our problem is to give a formal definition corresponding to intuition, and to prove that our "constructive ordinals" satisfy constructively all desirable properties.

The first step in Section 2 is to describe which these desirable properties are.

2 Linear orders associated to a set of indexors

We define in this section the structure of linear orders associated to a set $\mathfrak{F}$ of "indexors", $\mathfrak{F}$-orders for short.

2.1 Indexors

First we need a set $\mathfrak{F}$ of indexors. An indexor will be noted I, J, K, I', I'', J', I_a, I_b, etc.

An *indexor* is simply a set that will be used as a set of indices for the families we shall consider. In the sequel a "finitely enumerated subset of A" is always a subset of A defined à la Bishop by a map $\mathbb{N}_k \to A$. If A is discrete, a finitely enumerated subset of A is a nonempty detachable subset.

Properties of the indexor set $\mathfrak{F}$
We will assume that

- $\mathbb{N}$ *and the nonempty finite sets* $\mathbb{N}_k = \{\, n \in \mathbb{N} \mid n < k \,\}$ *(*$k > 0$*) are elements of* $\mathfrak{F}$*;*
- *any finitely enumerated subset of an element of* $\mathfrak{F}$ *is isomorphic*[2] *to an element of* $\mathfrak{F}$*;*
- *if* $J \in \mathfrak{F}$*, the set of finitely enumerated subsets of* J *is isomorphic to an element of* $\mathfrak{F}$*;*
- $\mathfrak{F}$ *is stable by disjoint unions indexed by* $\mathfrak{F}$*: we will denote by* $I + J$ *a disjoint union of* I *and* J*, and by* $\sum_{i \in I} J_i$ *a disjoint union of the family* $(J_i)_{i \in I}$*.*

Disjoint unions are to be understood as "direct sums" in the category of sets. Thus the disjoint union $J = \sum_{i \in I} J_i$ comes with a family $\iota_\ell : J_\ell \to J$ of injective maps realising J as the direct sum of the J_i's in the category of sets.

If we restrict ourselves to countable ordinals, we can take for $\mathfrak{F}$ the set

$$\boxed{\mathfrak{F}_2 = \{\, \mathbb{N}_k \mid k \in \mathbb{N}, k > 0 \,\} \cup \{\mathbb{N}\}}$$

with convenient operations for disjoint unions.

Any other indexor set $\mathfrak{F}$ will contain $\mathfrak{F}_2$.

An $\mathfrak{F}$-*indexed family of elements of* E is a family $(x_i)_{i \in I}$ where $I \in \mathfrak{F}$ and the x_i's $\in E$. The set of $\mathfrak{F}$-indexed families of elements of E is denoted by $\mathrm{Fam}(\mathfrak{F}, E)$.

We shall restrict the use of subscripts for ordinal variables to this meaning, and use superscripts for all other uses.

[2] In the category of sets.

2.2 Axioms

A structure of $\mathfrak{F}$-*order* is given as $(E, =, <, \leqslant, 0_E, \sup, \mathrm{succ})$, where

- $<$ and $\leqslant$ are binary relations defined on $(E, =)$;
- 0_E is an element of E and we let $E^\star = \{\, \alpha \in E \,|\, 0_E < \alpha \,\}$;
- sup is a map from $\mathrm{Fam}(\mathfrak{F}, E^\star)$ to $E^\star$: taking as input an element $(\alpha_i)_{i\in I}$ of $\mathrm{Fam}(\mathfrak{F}, E^\star)$, it constructs an element of $E^\star$ denoted by $\alpha = \sup_{i\in I} \alpha_i$;
- succ is a map from $\mathrm{Fam}(\mathfrak{F}, E)$ to $E^\star$: taking as input an element $(\alpha_i)_{i\in I}$ of $\mathrm{Fam}(\mathfrak{F}, E)$, it constructs an element of $E^\star$ denoted by $\alpha = \mathrm{succ}_{i\in I} \alpha_i$.

These data are to satisfy the following axioms.

Axioms for $\mathfrak{F}$-orders:

1. *(reflexivity and antisymmetry)* $\alpha = \beta$ *if and only if* $\alpha \leqslant \beta$ *and* $\beta \leqslant \alpha$;
2. $0_E \leqslant \alpha$;
3. *(irreflexivity) if* $\alpha < \alpha$, *then* $0_E = \beta$;
4. *if* $\alpha < \beta$, *then* $\alpha \leqslant \beta$;
5. *(transitivity 1) if* $\alpha \leqslant \beta$ *and* $\beta \leqslant \gamma$, *then* $\alpha \leqslant \gamma$;
6. *(transitivity 2) if* $\alpha < \beta$ *and* $\beta \leqslant \gamma$, *then* $\alpha < \gamma$;
7. *(transitivity 3) if* $\alpha \leqslant \beta$ *and* $\beta < \gamma$, *then* $\alpha < \gamma$;
8. *if* $\alpha < \mathrm{succ}(\beta)$, *then* $\alpha \leqslant \beta$;
9. *if* $\alpha < \gamma$ *and* $\beta < \gamma$, *then* $\sup(\alpha, \beta) < \gamma$;
10. *if* $\alpha < \sup(\alpha, \beta)$, *then* $\alpha < \beta$;
11. *if* $\gamma < \alpha$ *and* $\alpha \leqslant \sup(\beta, \gamma)$, *then* $\alpha \leqslant \beta$;
12. *(characteristic property of* succ*) for* $(\alpha_i)_{i\in I} \in \mathrm{Fam}(\mathfrak{F}, E)$ *and* $\beta \in E$ *we have*
$$\alpha_i < \beta \text{ for all } i \in I \quad \text{if and only if} \quad \mathrm{succ}_{i\in I}\, \alpha_i \leqslant \beta;$$
13. *(characteristic property of* sup*) for* $(\alpha_i)_{i\in I} \in \mathrm{Fam}(\mathfrak{F}, E^\star)$ *and* $\beta \in E$ *we have*
$$\alpha_i \leqslant \beta \text{ for all } i \in I \quad \text{if and only if} \quad \sup_{i\in I} \alpha_i \leqslant \beta;$$
14. *if* $\gamma < \beta$ *for all* $\gamma < \alpha$, *then* $\alpha \leqslant \beta$.
15. *either* $\alpha \leqslant 0_E$ *or* $0_E < \alpha$.

N.b.: $\mathrm{succ}(\alpha)$ denotes the succ of a family with only one element α, $\mathrm{succ}(\alpha, \beta)$ denotes the succ of a family with elements α and β, and $\sup(\alpha, \beta)$ denotes its sup.

The *category of* $\mathfrak{F}$-*orders* is defined by its morphisms

$$(E, =_E, <_E, \leqslant_E, 0_E, \sup_E, \mathrm{succ}_E) \longrightarrow (F, =_F, <_F, \leqslant_F, 0_F, \sup_F, \mathrm{succ}_F),$$

which are maps from E to F preserving the structure (with the usual meaning).

Comments.
1) Let $\gamma \in E$ and $(\alpha_n)_{n\in\mathbb{N}}$ such that $\alpha_n = \gamma$ or $\alpha_n = \mathrm{succ}(\gamma)$ for each n. The element $\mathrm{succ}_{n\in\mathbb{N}}\, \alpha_n$

hesitates between $\mathrm{succ}(\gamma)$ and $\mathrm{succ}(\mathrm{succ}(\gamma))$. Thus there is no hope that the disjunction "$\alpha \leqslant \beta$ or $\beta < \alpha$" be explicit for elements $\beta > 0_E$. Consequently, we have introduced the sup map together with its axioms in order to best describe in what sense the order can be thought of as "linear". Perhaps this is not optimal (reasonable axioms, satisfied in our set $\mathbf{Ord}_2$ of ordinals of the second class, might be missing).

2) If we had authorised the indexor $\mathbb{N}_0$, the element $\mathrm{succ}_{i\in\mathbb{N}_0}\,\alpha_i$ would necessarily have been equal to 0_E.

3) The irreflexivity is given a form that, instead of stating a negation, allows E to reduce to a singleton. This happens if and only if $0_E = \mathrm{succ}(0_E)$, which implies $0_E < 0_E$ using Axiom *12*.

4) Axiom *15* expresses that $\{0_E\}$ is detachable. This contrasts with the fact that elements $> 0_E$ do not define detachable singletons. We have defined sups on $E^\star$ rather than on E in order to satisfy constructively the disjunction of Axiom *15*. We remove this restriction for finite sups in Proposition 2.1. The need for infinite sups arises later, within ordinal arithmetic.

5) The characteristic property of sup shows that this law satisfies idempotence as well as generalised associativity and commutativity.

■

2.3 Some properties

Let us extend the sup map to finite lists in E.

Proposition and definition 2.1. *For $\alpha^1, \dots, \alpha^r \in E$ we let*

$$\sup(\alpha^1, \dots, \alpha^r) \stackrel{\text{def}}{=} \begin{cases} 0_E \text{ if } \alpha^1 = \cdots = \alpha^r = 0_E \\ \text{the sup of the } \alpha^k > 0_E \text{ otherwise.} \end{cases}$$

The characteristic property of upper bounds is satisfied for these finite sups:

$$\sup(\alpha^1, \dots, \alpha^r) \leqslant \beta \iff \alpha^1 \leqslant \beta \text{ and } \dots \text{ and } \alpha^r \leqslant \beta.$$

Fact 2.2. *Axioms 8 to 11 and 14 are in fact equivalences:*

8. $\alpha < \mathrm{succ}(\beta)$ *if and only if* $\alpha \leqslant \beta$*;*

9. $\alpha < \gamma$ *and* $\beta < \gamma$ *hold simultaneously if and only if* $\sup(\alpha, \beta) < \gamma$*;*

10. $\alpha < \sup(\alpha, \beta)$ *if and only if* $\alpha < \beta$ *;*

11. if $\gamma < \alpha$*, then* $\alpha \leqslant \sup(\beta, \gamma)$ *holds if and only if* $\alpha \leqslant \beta$*;*

14. $\alpha \leqslant \beta$ *if and only if* $\gamma < \beta$ *for all* $\gamma < \alpha$*.*

Proof. Use the transitivities and the characteristic property of sup.
For Axiom *8* we note that $\beta < \mathrm{succ}(\beta)$ follows from $\mathrm{succ}(\beta) \leqslant \mathrm{succ}(\beta)$ by the characteristic property of succ. □

Fact 2.3. *For any* $(\alpha_i)_{i\in J} \in \mathrm{Fam}(\mathfrak{F}, E)$*, we have* $\boxed{\mathrm{succ}_{i\in J}(\alpha_i) = \sup_{i\in J}(\mathrm{succ}(\alpha_i))}$.

Proof. Let $\alpha = \mathrm{succ}_{i\in J}(\alpha_i)$, $\beta^i = \mathrm{succ}(\alpha_i)$ for $i \in J$, $\gamma = \sup_{i\in J} \beta^i$. Then we have $\gamma \geqslant \beta^i > \alpha_i$ for each $i \in J$, and by the characteristic property of succ we get $\gamma \geqslant \alpha$.
It remains to show that $\alpha \geqslant \gamma$, i.e., that $\alpha \geqslant \beta^k$ for each $k \in J$, but $\beta^k = \mathrm{succ}(\alpha_k)$, and since $\alpha > \alpha_k$, we have $\alpha \geqslant \mathrm{succ}(\alpha_k)$. □

Remark 2.4. The previous result shows that we could define the structure using only the "infinitary" law $\sup : \mathrm{Fam}(\mathfrak{F}, E^\star) \to E^\star$ and the unary law $\mathrm{succ} : E \to E^\star$. In this case we have to modify Axiom *12* by stating only its unary version: "$\mathrm{succ}(\alpha) \leqslant \beta$ if and only if $\alpha < \beta$.

Fact 2.5. *We have* $\sup(\alpha,\beta) < \mathrm{succ}(\alpha,\beta)$ *and more precisely* $\mathrm{succ}(\alpha,\beta) = \mathrm{succ}(\sup(\alpha,\beta))$. *More generally,* $\boxed{\text{if } \alpha^k < \gamma \text{ for } k \in [\![1..r]\!], \text{ then } \sup(\alpha^1,\dots,\alpha^r) < \gamma}$, *and*

$$\boxed{\mathrm{succ}(\alpha^1,\dots,\alpha^r) = \mathrm{succ}(\sup(\alpha^1,\dots,\alpha^r))}.$$

Proof. Let $\gamma = \mathrm{succ}(\alpha,\beta)$. We have $\gamma > \alpha$ and $\gamma > \beta$, so that $\gamma > \sup(\alpha,\beta)$ by Axiom 9. By the characteristic property of succ we get $\gamma \geqslant \mathrm{succ}(\sup(\alpha,\beta))$. It remains to show that $\gamma \leqslant \mathrm{succ}(\sup(\alpha,\beta))$. In view of the definition of γ, we need to show that $\alpha < \mathrm{succ}(\sup(\alpha,\beta))$ and $\beta < \mathrm{succ}(\sup(\alpha,\beta))$. By transitivity this follows from $\mathrm{succ}(\sup(\alpha,\beta)) > \sup(\alpha,\beta)$, $\sup(\alpha,\beta) \geqslant \alpha$ and $\sup(\alpha,\beta) \geqslant \beta$. □

Fact 2.6. *Let* α,β *be elements of* E.

1. $\mathrm{succ}(\alpha) < \mathrm{succ}(\beta)$ *if and only if* $\alpha < \beta$.
2. $\mathrm{succ}(\alpha) \leqslant \mathrm{succ}(\beta)$ *if and only if* $\alpha \leqslant \beta$.

Proof. Use the characteristic property of $\mathrm{succ}(\alpha)$, transitivities and Axiom 8. □

We write $\boxed{F \subseteq_f I}$ in order to express that F is a finite, possibly empty list, of elements of I.

Fact 2.7. *Let* $\alpha,\beta^1,\dots,\beta^m \in E$.

1. *Assume that* $\alpha = \mathrm{succ}_{i\in J}\,\alpha_i$ *with* $(\alpha_i)_{i\in J} \in \mathrm{Fam}(\mathfrak{F},E)$ *and that* $\alpha_i < \sup(\beta^1,\dots,\beta^m)$ *for all* $i \in J$. *Then* $\alpha \leqslant \sup(\beta^1,\dots,\beta^m)$.
2. *Assume that* $\beta^k = \mathrm{succ}_{i\in J_k}(\beta^k)_i$ *with* $((\beta^k)_i)_{i\in J_k} \in \mathrm{Fam}(\mathfrak{F},E)$ *for* $k \in [\![1..m]\!]$. *Let* $F_1 \subseteq_f J_1,\dots,F_m \subseteq_f J_m$ *not all be empty. If*
$$\alpha \leqslant \sup_{k\in[\![1..m]\!],\ j\in F_k} (\beta^k)_j,$$
then $\alpha < \sup(\beta^1,\dots,\beta^m)$.

Proof. *1.* This is the definition of succ.
2. Suppose, e.g., that F_1 is nonempty. Then $\alpha \leqslant \sup_{j\in F_1}(\beta^1)_j < \beta^1 \leqslant \sup(\beta^1,\dots,\beta^m)$. The strict inequality comes from Fact 2.5 because β^1 is $>$ than all $(\beta^1)_j$'s. □

3 Inductive construction of ordinals

In Sections 3 and 4, the indexor set $\mathfrak{F}$ is fixed but seldom explicited.

We shall define a set of ordinals **Ord** (more precisely $\mathbf{Ord}_{\mathfrak{F}}$) and we shall prove that it is an initial object in the category of $\mathfrak{F}$-orders.

First we define a set **ord** of "names for $\mathfrak{F}$-indexed ordinals" by an inductive definition. The simplest inductive definition of an infinite set is that of $\mathbb{N}$: it is given an element 0 and a "successor" map $x \mapsto s(x) : \mathbb{N} \to \mathbb{N}$. The inductive definition of **ord** is very similar to that of $\mathbb{N}$. In $\mathbb{N}$ each element is either 0 or an $s(x)$ for an $x \in \mathbb{N}$. Similarly, in **ord**, each element is either $\underline{0}$ or the succ of an $\mathfrak{F}$-indexed family in **ord**; we denote by $\mathbf{ord}^\star$ the set of elements of this second type.

Definition 3.1. The set **ord** (more precisely $\mathbf{ord}_{\mathfrak{F}}$) is defined in an inductive way: it is to admit a distinguished element $\underline{0}$ and a map

$$\mathrm{succ} : \mathrm{Fam}(\mathfrak{F},\mathbf{ord}) \to \mathbf{ord}$$

N.b.: the only constraint in this inductive definition is that succ be indeed a map from $\mathrm{Fam}(\mathfrak{F},\mathbf{ord})$ to **ord**.

An element of **ord** will be called an *ordinal* in the sequel.

When $\mathfrak{F} = \mathfrak{F}_2$ we get the set of "names of countable ordinals", denoted by $\mathbf{ord}_2$.

Remark 3.2. Each element $\alpha \in \mathbf{ord}^\star$ is given with two data:

1. the indexor used in the definition of α: it will be denoted by In_α;
2. the element $\chi_{\mathbf{ord}}(\alpha, i)_{i \in \mathrm{In}_\alpha}$ of $\mathrm{Fam}(\mathfrak{F}, \mathbf{ord})$ such that $\alpha = \mathrm{succ}_{i \in \mathrm{In}_\alpha} \chi_{\mathbf{ord}}(\alpha, i)$.

Thus the inductive definition of **ord** implies the existence of a map $\alpha \mapsto \mathrm{In}_\alpha : \mathbf{ord}^\star \to \mathfrak{F}$ and the existence of a dependent family $(\alpha, i) \mapsto \chi_{\mathbf{ord}}(\alpha, i)$ which is defined for $\alpha \in \mathbf{ord}^\star$ and $i \in \mathrm{In}_\alpha$. In order to make the text more readable we will perform a slight abuse of notation: we shall not mention the construction of the dependent family $\chi_{\mathbf{ord}}$, and the notation α_i will be an abbreviation for $\chi_{\mathbf{ord}}(\alpha, i)$.

With these conventions we may write $\boxed{\alpha = \mathrm{succ}_{i \in \mathrm{In}_\alpha} \alpha_i}$.

For $\alpha^1, \ldots, \alpha^r \in \mathbf{ord}$ we define $\boxed{\mathrm{succ}(\alpha^1, \ldots, \alpha^r) = \mathrm{succ}_{i \in [\![1..r]\!]}(\alpha^i)}$.

In particular, if $\alpha \in \mathbf{ord}$, its *immediate successor* $\mathrm{succ}(\alpha)$ is the element $\beta = \mathrm{succ}_{i \in \mathrm{In}_\beta} \beta_i$ where $\mathrm{In}_\beta = \mathbb{N}_1 = \{0\}$ and $\beta_0 = \alpha$. The sequence $(\underline{m})_{m \in \mathbb{N}}$ in **ord** is defined inductively by $\underline{m+1} = \mathrm{succ}(\underline{m})$. Then we can define $\boxed{\omega = \mathrm{succ}_{n \in \mathbb{N}} \underline{n}}$.

In order to prove a property of $\alpha = \mathrm{succ}_{i \in \mathrm{In}_\alpha} \alpha_i$ it is sufficient to prove the property for each α_i. In a similar way we can construct inductively a map whose domain is **ord**, or define inductively a predicate on **ord**. This is given precisely in Fact 3.4.

3.1 Subordinals

Here is a correct inductive definition.

Definition 3.3. Let $\alpha = \mathrm{succ}_{i \in \mathrm{In}_\alpha} \alpha_i \in \mathbf{ord}^\star$. An element β of **ord** is said to be a *definitional subordinal* of α if $\beta = \alpha_i$ for an $i \in \mathrm{In}_\alpha$. We write this $\beta \lessdot_1 \alpha$. The element γ is a *subordinal of* α if it is a definitional subordinal of α or a subordinal of a definitional subordinal of α. We write this $\gamma \lessdot \alpha$.

Thus $\underline{0}$ is the only element of **ord** which has no subordinal.

Let us remark that the notion of subordinal is defined within the set **ord** and that it will not be possible to descend this notion to the quotient **Ord**.

Fact 3.4. *Relations $\lessdot_1$ and $\lessdot$ on* **ord** *are well-founded.*

Consequently there is no "infinite branch" in the "tree" of subordinals of an element of **ord**. Precisely:

Fact 3.5. *A sequence $(\alpha^j)_{j=1,2,\ldots}$ in* **ord**, *where each α^{j+1} is a subordinal of α^j, reaches in a finite number of steps $\alpha^r = \underline{0}$.*

Remark that in order to perform a construction (or a proof) by $\lessdot_1$-induction or by $\lessdot$-induction, the case $\underline{0}$ has to be dealt with separately since it has no subordinal. Nevertheless, until ordinal arithmetic page 15, we shall be able to avoid this case distinction.

3.2 Definition of the sup law

Definition 3.6.

1. The law $\mathrm{sup} : \mathrm{Fam}(\mathfrak{F}, \mathbf{ord}^\star) \to \mathbf{ord}^\star$ is defined in the following way.

   Let $(\alpha^j)_{j \in J}$ be a family in $\mathbf{ord}^\star$ with $J \in \mathfrak{F}$.

   If $\alpha^j = \mathrm{succ}_{i \in I_j} (\alpha^j)_i$, then $\boxed{\mathrm{sup}_{j \in J}(\alpha^j)}$ is the element $\varepsilon = \mathrm{succ}_{k \in K} \varepsilon_k$ where

   - K is the disjoint union of the In_{α^j}'s;
   - $(\varepsilon_k)_{k \in K}$ is the family defined by $\varepsilon_k = (\alpha^j)_i$ if $\iota_j(i) = k$

(Here $\iota_j : \mathrm{In}_{\alpha^j} \to K$ is the injective map from In_{α^j} to the disjoint union of the In_{α^j}'s).
We shall write $\sup_{j\in[\![1..r]\!]}(\alpha^j) = \sup(\alpha^1, \dots, \alpha^r)$.

2. Finite sups in **ord** are defined in the following way.

$$\sup(\alpha^1, \dots, \alpha^r) \stackrel{\text{def}}{=} \begin{cases} \underline{0} \text{ if } \alpha^1 = \cdots = \alpha^r = \underline{0} \\ \text{sup of the } \alpha^k \in \mathbf{ord}^\star \text{ otherwise.} \end{cases}$$

We note that Item 2 is formally included in Item 1 if we adopt the convention $\mathrm{In}_{\underline{0}} = \mathbb{N}_0$. However, this convention would not allow us to define arbitrary $\mathfrak{F}$-indexed sups in **ord**.

3.3 Definition of $\leqslant$ and of $<$

The main job remains to be done, i.e., to define two binary relations $\leqslant$ and $<$ on **ord** with the wanted properties, i.e.

- the relation "$\alpha \leqslant \beta$ and $\beta \leqslant \alpha$" has to be an equivalence relation (we shall denote by **Ord** the quotient set),
- the relations $\leqslant$ and $<$ and the maps sup and succ must descend to the quotient (we shall not change their names),
- with these maps and relations, **Ord** must be an $\mathfrak{F}$-order.

For this, we define inductively two asymmetric relations between, on the left side, an element of **ord** and, on the right side, *a nonempty finite list (up to permutation)*[3] of elements of **ord**:

$$\boxed{\alpha \leqslant \beta^1, \dots, \beta^m \quad \text{and} \quad \alpha < \beta^1, \dots, \beta^m \quad (m \geqslant 1)\ .}$$

We shall see in Lemma 3.19 that their meanings are $\alpha \leqslant \sup(\beta^1, \dots, \beta^m)$ et $\alpha < \sup(\beta^1, \dots, \beta^m)$.

Conventions.
Letters α, β, γ, ε, possibly with exponents, indices or primes, are used for elements of **ord**.
If α is an element of **ord** and if F is a finite list, possibly empty, in In_α, we denote by α_F the list of α_i's with i in F.

The two relations are defined by simultaneous induction in the following way.

The order in the list of the β^j's does not matter.
Particular cases involving $\underline{0}$ are avoided by using the convention $\mathrm{In}_{\underline{0}} = \mathbb{N}_0$,
but $\mathbb{N}_0$ is not an element of $\mathfrak{F}$.

$\alpha \leqslant \beta^1, \dots, \beta^m \ (m \geqslant 1)$ means $\alpha_i < \beta^1, \dots, \beta^m$ for all $i \in \mathrm{In}_\alpha$.

$\alpha < \beta^1, \dots, \beta^m \ (m \geqslant 1)$ means there are $F_1 \subseteq_f \mathrm{In}_{\beta^1}, \dots, F_m \subseteq_f \mathrm{In}_{\beta^m}$ not all empty such that $\alpha \leqslant \beta^1_{F_1}, \dots, \beta^m_{F_m}$.

These definitions are correct since elements of **ord** are inductively defined and the pair of definitions is inductive.

Without the convention that $\mathrm{In}_{\underline{0}} = \mathbb{N}_0$, we would have had to include Fact 3.7 in the definition. This convention is "a little miracle" allowing us to avoid a case by case reasoning w.r.t. the disjunction "$\alpha = \underline{0}$ or $\alpha \in \mathbf{ord}^\star$" in the proofs.

The relation $\alpha =_{\mathbf{Ord}} \beta$ is defined as meaning "$\alpha \leqslant \beta$ and $\beta \leqslant \alpha$".

[3] One also calls this a nonempty *multiset*.

We shall show in Section 4 the relation $\cdot =_{\mathbf{Ord}} \cdot$ is an equivalence relation and we shall define the set **Ord** as the quotient of **ord** by this relation.

Let us note that until Theorem 4.8, the symbol $=$ between two elements of **ord** is the equality in **ord** and has not the meaning of $=_{\mathbf{Ord}}$. Nevertheless, after having shown that the relations and the laws of **ord** descend to the quotient **Ord**, the statements with the symbol $=$ will also work for the symbol $=_{\mathbf{Ord}}$.

3.4 Finite ordinals, bounded ordinals

We start with a few properties of $\underline{0}$.

Fact 3.7. *Let m be an integer $\geqslant 1$, $\alpha, \beta^1, \dots, \beta^m \in \mathbf{ord}$, and $\gamma \in \mathbf{ord}^\star$. We have*

1. $\underline{0} \leqslant \beta^1, \dots, \beta^m$;
2. $\underline{0} < \gamma, \beta^2, \dots, \beta^m$;
3. $\alpha < \underbrace{\underline{0}, \dots, \underline{0}}_{m \text{ times}}$ *is impossible.*

Proof. Straightforward from the definitions. □

Remark 3.8. Axiom *15* will be valid in **Ord** because every element of **ord** is given either as $\underline{0}$ or as an element of $\mathbf{ord}^\star$, always > 0 by Item *2* of 3.7.

Fact 3.9. *Let $m, n \in \mathbb{N}$. Then*

1. *$m \leqslant n$ if and only if $\underline{m} \leqslant \underline{n}$;*
2. *$m < n$ if and only if $\underline{m} < \underline{n}$;*
3. *$\underline{m} \leqslant \underline{n}$ and $\underline{n} < \underline{m}$ are incompatible.*

Proof. Concerning the direct implications in *1* and *2*, we write $n = m + r$ and we do an induction on r. For the reverse implications, cases $m = 0$ and $n = 0$ are already known. Next we see that $\underline{m+1} \leqslant \underline{n+1}$ implies $\underline{m} \leqslant \underline{n}$, and that $\underline{m+1} < \underline{n+1}$ implies $\underline{m} < \underline{n}$. This allows us to conclude by induction on m.
Item *3* follows from items *1* and *2*. □

An element $\alpha \in \mathbf{ord}$ is said to be *finite* if $\alpha =_{\mathbf{Ord}} \underline{m}$ for an $m \in \mathbb{N}$, *bounded* if $\alpha \leqslant \underline{m}$ for an $m \in \mathbb{N}$. Bounded ordinals are much more complicate than finite ordinals (see examples 3.20 and 3.21).

3.5 In classical mathematics

Proposition 3.10 shows that the law of excluded middle (LEM) dramatically simplifies and/or obscures what is the set $\mathbf{ord}_{\mathfrak{F}}$.

Proposition 3.10. *Assume LEM. Then for $\alpha, \beta \in \mathbf{ord}$, we have $\alpha \leqslant \beta$ or $\beta < \alpha$. Moreover, if $\beta < \alpha$, there exists an $i \in \mathrm{In}_\alpha$ such that $\beta \leqslant \alpha_i$.*

Proof. We prove by simultaneous induction the two following properties.

$$\text{“}\alpha \leqslant \beta \text{ or } \beta < \alpha\text{”} \quad \text{and} \quad \text{“}\beta \leqslant \alpha \text{ or } \alpha < \beta\text{”}.$$

By induction hypothesis, we have for all $i \in \mathrm{In}_\alpha$ and all $j \in \mathrm{In}_\beta$, “$\alpha \leqslant \beta_j$ or $\beta_j < \alpha$”, and also “$\beta \leqslant \alpha_i$ or $\alpha_i < \beta$”.
The first disjunction implies by LEM that either $\beta_j < \alpha$ for all $j \in \mathrm{In}_\beta$ or there is $j \in \mathrm{In}_\beta$ such that $\alpha \leqslant \beta_j$. In the first case, we have $\beta \leqslant \alpha$ by definition of $\cdot \leqslant \cdots$. In the second case, we have $\alpha < \beta$ by definition of $\cdot < \cdots$, with for $F \subseteq_f \mathrm{In}_\beta$ the list $[\,j\,]$.
Symmetric reasoning for the second disjunction. □

N.b.: for countable ordinals, the limited principle of omniscience (**LPO**) is sufficient for proving the proposition.

Corollary 3.11. *Assume LEM. Any ordinal $\alpha > 0$ is either an immediate successor, or the upper bound of ordinals $\gamma < \alpha$.*

Proof. Consider $\alpha = \mathrm{succ}_{i \in \mathrm{In}_\alpha} \alpha_i$ and compare α with $\sup_{i \in \mathrm{In}_\alpha} \alpha_i$. The details are left to the reader. □

Corollary 3.12. *Assume LEM. Any bounded ordinal is finite.*

Proof. Left to the reader: use Fact 3.15. □

3.6 First consequences

The following fact shows that the succ law will satisfy the characteristic property given in Axiom *12* when we shall know that it descends to the quotient **Ord**.

Fact 3.13. `succdef`. *We have $\alpha \leqslant \beta$ if and only if $\alpha_i < \beta$ for all $i \in \mathrm{In}_\alpha$.*

Proof. This property is tautological: this is the definition of $\alpha \leqslant \beta$. □

Similarly, the following fact shows that the sup law will satisfy the characteristic property given in Axiom *13* when we shall know that it descends to the quotient **Ord**.

Fact 3.14. `supdef`. *Let $(\alpha^j)_{j \in J}$ be a family in $\mathbf{ord}^\star$ with $J \in \mathfrak{F}$, $\gamma = \sup_{j \in J}(\alpha^j)$, and $\beta \in \mathbf{ord}$. We have $\gamma \leqslant \beta$ if and only if $\alpha^j \leqslant \beta$ for all $j \in J$. In particular, if $\sup(\alpha, \beta) \leqslant \beta$, then $\alpha \leqslant \beta$. N.b.: the result is equally true for finite sups in* **ord**.

Proof. Another linguistic tautology. We have $\alpha^j = \mathrm{succ}_{i \in I_j}(\alpha^j)_i$ for an $I_j \in \mathfrak{F}$. By definitions of γ and of $\leqslant$, the inequality $\gamma \leqslant \beta$ means that for each $j \in J$ and each $i \in I_j$ we have $(\alpha^j)_i < \beta$, i.e., that for each $j \in J$ we have $\alpha^j \leqslant \beta$. □

The following fact shows that Axiom *8* will be valid when we shall descend to the quotient **Ord**.

Fact 3.15. `ax8`. *We have $\alpha < \mathrm{succ}(\beta)$ if and only if $\alpha \leqslant \beta$.*

Proof. Recall that the element $\gamma = \mathrm{succ}(\beta)$ is defined by $\mathrm{In}_\gamma = \{0\}$ and $\gamma_0 = \beta$. By definition, $\alpha < \gamma$ means that $\alpha \leqslant \gamma_F$ for a nonempty list $F \subseteq_f \{0\}$. This forces $F = [\,0\,]$ and $\gamma_F = \beta$. Thus, better than an equivalence, this is a tautology. □

The following fact will allow us to shorten certain proofs by induction.

Fact 3.16.

a. *We have an inequality $\alpha \leqslant \beta$ if and only if for each $i \in \mathrm{In}_\alpha$, there exists a nonempty $F_i \subseteq_f \mathrm{In}_\beta$ such that $\alpha_i \leqslant \beta_{F_i}$.*

b. *We have an inequality $\alpha < \beta$ if and only if there exists a nonempty $F \subseteq_f \mathrm{In}_\beta$ such that for each $i \in \mathrm{In}_\alpha$ we have $\alpha_i < \beta_F$.*

Proof. Straightforward from the definitions. □

Now we let behind tautological proofs and turn to inductive proofs.

Fact 3.17.

1. `wkn`. *(weakening) if* $\alpha \leqslant \beta^1, \dots, \beta^m$*, then for each* β*, we have* $\alpha \leqslant \beta, \beta^1, \dots, \beta^m$.

2. `ctn`. *(contraction) if* $\alpha \leqslant \beta^1, \beta^1, \beta^2, \dots, \beta^m$*, then* $\alpha \leqslant \beta^1, \beta^2, \dots, \beta^m$

3. *Same properties with* $<$ *instead of* $\leqslant$.

Proof. Proofs by induction, applying the definitions. □

The following lemma is a corollary of Fact 3.16. Item *1* (resp. *2*) will imply that the succ (resp. sup) map descends to the quotient in **Ord** (resp. $\mathbf{Ord}^\star$). Item *3* will imply that the relations $\leqslant$ and $=$ are reflexive in **Ord**; Items *5* and *7* will imply Axioms *3* and *14* for **Ord**.

Lemma 3.18.

1. `succ0`. *Let* $\alpha, \beta \in \mathbf{ord}$ *with* $\mathrm{In}_\alpha = \mathrm{In}_\beta$ *and* $\alpha_i \leqslant \beta_i$ *for all* $i \in \mathrm{In}_\alpha$*. Then* $\alpha \leqslant \beta$.

2. `sup0`. *Let* $\alpha, \beta \in \mathbf{ord}^\star$ *with* $\mathrm{In}_\alpha = \mathrm{In}_\beta$ *and* $\alpha_i \leqslant \beta_i$ *for all* $i \in \mathrm{In}_\alpha$*. Then*
$$\sup_{i \in \mathrm{In}_\alpha} \alpha_i \leqslant \sup_{i \in \mathrm{In}_\beta} \beta_i.$$
The result works also for finite sups in **ord**.

3. `rfl`. *For all* $\alpha \in \mathbf{ord}$*, we have* $\alpha \leqslant \alpha$*. A fortiori,* $\alpha \leqslant \alpha, \beta^1, \dots, \beta^m$.

4. `succ1`. *For all* $\alpha \in \mathbf{ord}^\star$ *and all* $i \in \mathrm{In}_\alpha$*, we have* $\alpha_i < \alpha$*. A fortiori,* $\alpha_i < \alpha, \beta^1, \dots, \beta^m$.

5. `irfl`. *For all* $\alpha \in \mathbf{ord}$*,* $\alpha < \alpha$ *is impossible.*

6. $\alpha < \mathrm{succ}(\alpha)$.

7. `ax14`. *if* $\gamma < \beta$ *for all* $\gamma < \alpha$ *then* $\alpha \leqslant \beta$.

Proof. *1.* Straightforward from 3.16*a*. We take $F = \{i\}$.

2. Elements $\gamma = \sup_{i \in J} \alpha_i$ and $\epsilon = \sup_{i \in J} \beta_i$ satisfy the hypotheses of Item *1*.

3. By induction: we use 3.16*a*., we take $F = \{i\}$ and $\alpha \leqslant \alpha$ reduces to $\alpha_i \leqslant \alpha_i$.

4. By induction: we use 3.16*b*., we take $F = \{i\}$ and $\alpha_i < \alpha$ reduces to $(\alpha_i)_j < \alpha_i$.

5. By induction: we use 3.16*b*., we take $F = \{i\}$ and "$\alpha < \alpha$ is impossible" reduces to: "$\alpha_i < \alpha_i$ is impossible".

6. Apply `succ1` to $\beta = \mathrm{succ}(\alpha)$.

7. If $\alpha = \underline{0}$, the conclusion is clear. If $\alpha = \mathrm{succ}_{i \in I} \alpha_i$, as $\alpha_i < \alpha$ for each $i \in \mathrm{In}_\alpha$ (Item *4*), the hypothesis that $\gamma < \beta$ for all $\gamma < \alpha$ shows that $\alpha_i < \beta$ for all $i \in \mathrm{In}_\alpha$. We conclude by 3.13 that $\alpha \leqslant \beta$. □

Lemma 3.19. *We have* $\alpha \leqslant \beta^1, \dots, \beta^m$ *if and only if* $\alpha \leqslant \sup(\beta^1, \dots, \beta^m)$.
Similarly, we have $\alpha < \beta^1, \dots, \beta^m$ *if and only if* $\alpha < \sup(\beta^1, \dots, \beta^m)$.

Proof. Let us write
$$\alpha \preccurlyeq \beta^1, \dots, \beta^m \text{ for } \alpha \leqslant \sup(\beta^1, \dots, \beta^m) \quad \text{and} \quad \alpha \prec \beta^1, \dots, \beta^m \text{ for } \alpha < \sup(\beta^1, \dots, \beta^m).$$
Now we see that $\cdot \preccurlyeq \cdots$ and $\cdot \prec \cdots$ satisfy the same inductive properties as the ones which are used to define $\cdot \leqslant \cdots$ and $\cdot < \cdots$. □

Example 3.20. Let $(v_n)_{n\in\mathbb{N}}$ be a sequence in $\{0,1\}$ which takes at most once the value 1. The lesser limited principle of omniscience **LLPO** says that we have

$$\exists k \in \{0,1\}\ \forall n\ \ (v_n = 1 \Rightarrow\ n \equiv k \bmod 2). \tag{$*$}$$

From such a sequence $(v_n)_{n\in\mathbb{N}}$ let us define ε, ε^1 and $\varepsilon^2 \in$ **ord** in the following way:

$$\varepsilon = \mathrm{succ}_{n\in\mathbb{N}}\ \underline{v_n}, \quad \varepsilon^1 = \mathrm{succ}_{m\in\mathbb{N}}\ \underline{v_{2m}}, \quad \varepsilon^2 = \mathrm{succ}_{m\in\mathbb{N}}\ \underline{v_{2m+1}}.$$

Then we have $\varepsilon \leqslant \sup(\varepsilon^1, \varepsilon^2)$. But $\varepsilon \leqslant \varepsilon^1$ gives $k = 0$ in $(*)$ and $\varepsilon \leqslant \varepsilon^2$ gives $k = 1$ in $(*)$. Thus, the disjunction $\varepsilon \leqslant \varepsilon^1$ or $\varepsilon \leqslant \varepsilon^2$ has no constructive proof. Assuming the disjunction for an arbitrary (v_n) would imply **LLPO**.

Example 3.21. Let $(u_n)_{n\in\mathbb{N}}$ be a nondecreasing sequence in $\{0,1\}$. The principle of omniscience **LPO** says that such a sequence is eventually constant:

$$\exists n \in \mathbb{N}\ \forall m \in \mathbb{N},\ u_m \leqslant u_n. \tag{$*$}$$

From such a sequence $(u_n)_{n\in\mathbb{N}}$ let us define ε and $\varepsilon' \in$ **ord** in the following way:

$$\varepsilon = \mathrm{succ}_{n\in\mathbb{N}}\ \underline{u_n}, \qquad \varepsilon' = \mathrm{succ}_{n\in\mathbb{N}}\ \underline{1+u_n}.$$

We notice that the strict inequality $\varepsilon < \varepsilon'$ is equivalent to (using 3.9, 3.16 and 3.19)

$$\exists n \in \mathbb{N}\ \forall m \in \mathbb{N},\ u_m < 1 + u_n,$$

which is the same thing as $(*)$. In fact, ε hesitates between 1 and 2, ε' hesitates between 2 and 3 and the inequality $\varepsilon < \varepsilon'$ is valid if we assume **LPO**. But asserting $\varepsilon < \varepsilon'$ for all sequences (u_n) implies **LPO** in constructive mathematics. Here we see that hesitating between 1 and 2 for an infinite sequence has the same flavor as (in a classical setting) hesitating between bounded and unbounded for an infinite sequence of natural numbers: adding 1 to each term of the sequence increases strictly the sup only if the sequence is bounded.

4 Fundamental results

4.1 $\mathbf{Ord}_{\mathfrak{F}}$ is an initial object in the category of $\mathfrak{F}$-orders

Lemma 4.1. *For $\alpha^1, \ldots, \alpha^r$ in* **ord** *($r \geqslant 1$), we have* $\boxed{\sup_{j\in[\![1..r]\!]} \alpha^j < \mathrm{succ}_{j\in[\![1..r]\!]} \alpha^j}$.

Proof. E.g., let us show that $\epsilon = \sup(\alpha,\beta) < \gamma = \mathrm{succ}(\alpha,\beta)$. We have $\mathrm{In}_\epsilon = \mathrm{In}_\alpha + \mathrm{In}_\beta$, with $\epsilon_k = \alpha_i$ if $\iota_1(i) = k$, and $\epsilon_k = \beta_j$ if $\iota_2(j) = k$. We have $\mathrm{In}_\gamma = \{1,2\}$ with $\gamma_1 = \alpha$ and $\gamma_2 = \beta$. We apply Fact 3.16*b*. with $F = \{1,2\}$. For an arbitrary k in In_ϵ, we have $\epsilon_k < \alpha, \beta$ since ϵ_k is α_i or β_j, and by `succ1`, we have $\alpha_i < \alpha$ (a fortiori $\alpha_i < \alpha, \beta$) and $\beta_j < \beta$ (a fortiori $\beta_j < \alpha, \beta$). □

Let us note that the preceding proof rely on the fact that definitions of $\leqslant$ and $<$ have been given with lists on the right hand side.

Lemma 4.2. (transitivities)

1. `trans1`. *If $\alpha \leqslant \beta^1, \ldots, \beta^m$ and, for each $j \in [\![1..m]\!]$, $\beta^j \leqslant \gamma^1, \ldots, \gamma^r$, then $\alpha \leqslant \gamma^1, \ldots, \gamma^r$.*
2. `trans2`. *If $\alpha < \beta^1, \ldots, \beta^m$ and, for each $j \in [\![1..m]\!]$, $\beta^j \leqslant \gamma^1, \ldots, \gamma^r$, then $\alpha < \gamma^1, \ldots, \gamma^r$.*
3. `trans3`. *If $\alpha \leqslant \beta^1, \ldots, \beta^m$ and, for each $j \in [\![1..m]\!]$, $\beta^j < \gamma^1, \ldots, \gamma^r$, then $\alpha < \gamma^1, \ldots, \gamma^r$.*

As particular cases:

- *if $\alpha \leqslant \beta$ and $\beta \leqslant \gamma$ then $\alpha \leqslant \gamma$;*
- *if $\alpha < \beta$ and $\beta \leqslant \gamma$ then $\alpha < \gamma$;*

- *if $\alpha \leqslant \beta$ and $\beta < \gamma$ then $\alpha < \gamma$.*

Proof. The three transitivities are being proved by simultaneous induction.

In order to prove `trans1`, we note that the hypothesis means that we have $\alpha_i < \beta^1, \ldots, \beta^m$ for all $i \in \mathrm{In}_\alpha$. Let us fix such an i. We use `trans2` with this α_i instead of α and we get $\alpha_i < \gamma^1, \ldots, \gamma^r$. Since this works for all $i \in \mathrm{In}_\alpha$, this gives the desired conclusion $\alpha \leqslant \gamma^1, \ldots, \gamma^r$.

In order to prove `trans2`, we note that the hypothesis implies that there are $G_j \subseteq_f \mathrm{In}_{\beta^j}$ not all empty such that $\alpha \leqslant \beta^1_{G_1}, \ldots, \beta^m_{G_m}$. We have also for $j \in [\![1..m]\!]$ and for all $h \in \mathrm{In}_{\beta^j}$, $\beta^j_h < \gamma^1, \ldots, \gamma^r$. A fortiori, this is true for the h's $\in G_j$. We use `trans3` with these β^j_h's instead of the β^j's. This gives the desired conclusion $\alpha < \gamma^1, \ldots, \gamma^r$.

In order to prove `trans3`, we note that the hypothesis implies (by weakening) that there are $F_k \subseteq_f \mathrm{In}_{\gamma^k}$ not all empty such that $\beta^j \leqslant \gamma^1_{F_1}, \ldots, \gamma^r_{F_r}$ for $j \in [\![1..m]\!]$. This time we use `trans1` with the γ^k_ℓ's instead of the γ^k's and we deduce that $\alpha \leqslant \gamma^1_{F_1}, \ldots, \gamma^r_{F_r}$, which implies $\alpha < \gamma^1, \ldots, \gamma^r$. □

The following lemma shows that when descending to the quotient, Axiom *4* will be valid in **Ord**.

Lemma 4.3. `ax4`. *Let $\alpha, \beta^1, \ldots, \beta^m \in$ **ord**. If $\alpha < \beta^1, \ldots, \beta^m$, then $\alpha \leqslant \beta^1, \ldots, \beta^m$.*

Proof. Proof by induction on α. We have $\alpha < \beta^1, \ldots, \beta^m$ if and only if we can find $F_k \subseteq_f \mathrm{In}_{\beta^k}$ not all empty such that, for each $i \in \mathrm{In}_\alpha$, we have $\alpha_i \leqslant \beta^1_{F_1}, \ldots, \beta^m_{F_m}$. Let us fix an $i \in \mathrm{In}_\alpha$. For $j \in F_k$, we have $\beta^k_j < \beta^k$, and by weakening $\beta^k_j < \beta^1, \ldots, \beta^m$. By `trans3`, we get $\alpha_i < \beta^1, \ldots, \beta^m$. Finally, since this is true for all $i \in \mathrm{In}_\alpha$, we have $\alpha \leqslant \beta^1, \ldots, \beta^m$. □

The following fact shows that Axiom *9* will be valid when we shall descend to the quotient **Ord**.

Lemma 4.4. `ax9`. *If $\alpha < \gamma$ and $\beta < \gamma$, then $\sup(\alpha, \beta) < \gamma$.*

Proof. By definition, we have $\mathrm{succ}(\alpha, \beta) \leqslant \gamma$. Lemma 4.1 gives $\sup(\alpha, \beta) < \mathrm{succ}(\alpha, \beta)$. By transitivity, we get $\sup(\alpha, \beta) < \gamma$. □

Lemma 4.5. *Let n be a positive integer and $\alpha^1, \ldots, \alpha^n \in$ **ord**. It is impossible that, for each $i \in [\![1..n]\!]$, we have $\alpha^i < \alpha^1, \ldots, \alpha^n$.*

Proof. By induction. Using weakening, the hypothesis gives finite lists

$$F_1 \subseteq_f \mathrm{In}_{\alpha_1}, \ldots, F_n \subseteq_f \mathrm{In}_{\alpha_n}$$

not all empty, such that

$$\alpha^i \leqslant \alpha^1_{F_1}, \ldots, \alpha^n_{F_n} \text{ for } i \in [\![1..m]\!].$$

In particular, for $j \in F_i$ (if F_i is nonempty) we have

$$\alpha^i_j < \alpha^1_{F_1}, \ldots, \alpha^n_{F_n}.$$

This reduces to the induction hypothesis with the nonempty list $\alpha^1_{F_1}, \ldots, \alpha^n_{F_n}$ instead of the list $\alpha^1, \ldots, \alpha^n$. □

Lemma 4.6. *Let $\alpha^1, \ldots, \alpha^n, \beta^1, \ldots, \beta^m \in$ **ord** $(n, m \geqslant 1)$.*

1. *If $\alpha^i < \alpha^1, \ldots, \alpha^n, \beta^1, \ldots, \beta^m$ for $i \in [\![1..n]\!]$, then $\alpha^i < \beta^1, \ldots, \beta^m$ for each i.*
2. *Let $F_1 \subseteq_f \mathrm{In}_{\alpha_1}, \ldots, F_n \subseteq_f \mathrm{In}_{\alpha_n}$. If $\alpha^i \leqslant \alpha^1_{F_1}, \ldots, \alpha^n_{F_n}, \beta^1, \ldots, \beta^m$ for $i \in [\![1..n]\!]$, then $\alpha^i \leqslant \beta^1, \ldots, \beta^m$ for each i.*

Proof. 1. The hypothesis yields finite lists, not all empty,

$$F_1 \subseteq_f \mathrm{In}_{\alpha^1}, \ldots, F_n \subseteq_f \mathrm{In}_{\alpha^n}, G_1 \subseteq_f \mathrm{In}_{\beta^1}, \ldots, G_m \subseteq_f \mathrm{In}_{\beta^m}$$

such that

$$\alpha^i \leqslant \alpha^1_{F_1}, \ldots, \alpha^n_{F_n}, \beta^1_{G_1}, \ldots, \beta^m_{G_m} \text{ for } i \in [\![1..n]\!]. \tag{$*$}$$

Thus we have for $i \in [\![1..n]\!]$ and $j \in \mathrm{In}_{\alpha^i}$

$$\alpha^i_j < \alpha^1_{F_1}, \ldots, \alpha^i_{F_i}, \ldots, \alpha^n_{F_n}, \beta^1_{G_1}, \ldots, \beta^m_{G_m}$$

and a fortiori, with $F'_i = F_i \cup \{j\}$

$$\alpha^i_j < \alpha^1_{F_1}, \ldots, \alpha^i_{F'_i}, \ldots, \alpha^n_{F_n}, \beta^1_{G_1}, \ldots, \beta^m_{G_m}.$$

We have also by weakening, for $k \in [\![1..n]\!]$ and $\ell \in F_k$

$$\alpha^k_\ell < \alpha^1_{F_1}, \ldots, \alpha^i_{F'_i}, \ldots, \alpha^n_{F_n}, \beta^1_{G_1}, \ldots, \beta^m_{G_m}.$$

Thus by induction $\alpha^i_j < \beta^1_{G_1}, \ldots, \beta^m_{G_m}$. And since j is arbitrary, we get $\alpha^i \leqslant \beta^1_{G_1}, \ldots, \beta^m_{G_m}$ for $i \in [\![1..n]\!]$. This gives the desired conclusion, $\alpha^i < \beta^1, \ldots, \beta^m$ for $i \in [\![1..n]\!]$, if at least one list G_k is nonempty. If this is not the case, $(*)$ yields $\alpha^i \leqslant \alpha^1_{F_1}, \ldots, \alpha^n_{F_n}$ for $i \in [\![1..n]\!]$, with lists F_i not all empty. By definition, this implies $\alpha^i < \alpha^1, \ldots, \alpha^n$ for $i \in [\![1..n]\!]$, which is impossible by Lemma 4.5.

2. We have for $i \in [\![1..n]\!]$ and $j \in \mathrm{In}_{\alpha^i}$

$$\alpha^i_j < \alpha^1_{F_1}, \ldots, \alpha^i_{F_i}, \ldots, \alpha^n_{F_n}, \beta^1, \ldots, \beta^m$$

and a fortiori, with $F'_i = F_i \cup \{j\}$,

$$\alpha^i_j < \alpha^1_{F_1}, \ldots, \alpha^i_{F'_i}, \ldots, \alpha^n_{F_n}, \beta^1, \ldots, \beta^m.$$

We have also by weakening, for $k \in [\![1..n]\!]$ and $\ell \in F_k$

$$\alpha^k_\ell < \alpha^1_{F_1}, \ldots, \alpha^i_{F'_i}, \ldots, \alpha^n_{F_n}, \beta^1, \ldots, \beta^m.$$

Item *1* then yields $\alpha^i_j < \beta^1, \ldots, \beta^m$. As j is arbitrary, we get what we wanted: $\alpha^i \leqslant \beta^1, \ldots, \beta^m$ for $i \in [\![1..n]\!]$. □

The following fact shows that Axioms *10* and *11* will be valid when we shall descend to the quotient **Ord**.

Lemma 4.7.

1. `ax10`. *If $\alpha < \sup(\alpha, \beta)$, then $\alpha < \beta$;*
2. `ax11`. *If $\gamma < \alpha$ and $\alpha \leqslant \sup(\beta, \gamma)$, then $\alpha \leqslant \beta$.*

Proof. 1. Assume $\alpha < \sup(\alpha, \beta)$. Lemma 3.19 gives $\alpha < \alpha, \beta$. Item *1* of Lemma 4.6 gives $\alpha < \beta$.

2. Assume $\gamma < \alpha$ and $\alpha \leqslant \sup(\beta, \gamma)$. The first hypothesis gives $\gamma \leqslant \alpha_F$ for a nonempty $F \subseteq_f \mathrm{In}_\alpha$. The second hypothesis gives $\alpha \leqslant \gamma, \beta$ (by Lemma 3.19). By transitivity we have $\alpha \leqslant \alpha_F, \beta$. Item *2* of Lemma 4.6 gives $\alpha \leqslant \beta$. □

Theorem 4.8. *We have constructed* **Ord** *as an $\mathfrak{F}$-order.*

Proof. Using `rfl` and `trans1`, we first show that the equality is indeed an equivalence relation, and second that the relation $\leqslant$ descends to the quotient in **Ord**.

Similarly, `trans2` and `trans3` imply that the relation $<$ descends to the quotient in **Ord**.

The sup map descends to the quotient by Lemma 3.18, Item *1*.

The succ map descends to the quotient by Lemma 3.18, Item *2*.

It remains to note that all axioms of $\mathfrak{F}$-orders were proved above: see Fact 3.7 (Item *1*), Remark 3.8, Facts 3.13, 3.14. 3.15, Lemmas 3.18 (Items *3*, *5* and *7*), 4.2, 4.3, 4.4 and 4.7. □

The following theorem generalises Fact 3.9.

Theorem 4.9. *The set* **Ord** *is not reduced to a point. More precisely:*

- *for all* $\alpha, \beta \in \mathbf{Ord}$, $\beta \leqslant \alpha$ *and* $\alpha < \beta$ *are incompatible;*
- *the map* $n \mapsto \underline{n} : \mathbb{N} \to \mathbf{Ord}$ *is injective* ($m < n$ *if and only if* $\underline{m} < \underline{n}$).
- *for all* $\alpha \in \mathbf{Ord}$ *and* $n > m$ *in* $\mathbb{N}$, *it is impossible that* $\mathrm{succ}^{(n)}(\alpha) =_{\mathbf{Ord}} \mathrm{succ}^{(m)}(\alpha)$;

Proof. The first item is a consequence of `irfl` and of `trans2`. The rest follows. □

Theorem 4.10. **Ord** *is an initial object in the category of* $\mathfrak{F}$*-orders.*

Sketch of proof. The structure is "purely algebraic" and in order to construct **Ord**, we have only used the axioms of the structure.
In fact, let us consider an object $(E, =, <, \leqslant, 0_E, \sup, \mathrm{succ})$ in the category. Elements of **ord** do have their "copies" in E. Furthermore, the relations $\cdot < \cdots$ and $\cdot \leqslant \cdots$ defined in **ord** are valid in E by Fact 2.7 if interpreted in E with finite sups on the right hand side (as we may by Lemma 3.19). This implies that there is a unique morphism from **Ord** to E in the category. □

4.2 More properties

Proposition 4.11. *The binary relation* $<$ *on* **Ord** *is well-founded.*

Proof. Direct consequence of Fact 3.4. □

Lemma 4.12 (weak forms of the disjunction "$\alpha \leqslant \beta$ or $\beta < \alpha$")**.** *Let* $r \geqslant 1$ *and* $\alpha, \beta^1, \ldots, \beta^r, \gamma \in$ **ord**.

1. *If* $\gamma \leqslant \alpha$ *and* $\alpha < \gamma, \beta^1, \ldots, \beta^r$, *then* $\alpha < \beta^1, \ldots, \beta^r$.
2. *If* $\gamma < \alpha$ *and* $\alpha \leqslant \gamma, \beta^1, \ldots, \beta^r$, *then* $\alpha \leqslant \beta^1, \ldots, \beta^r$.

Proof. Introduce $\beta = \sup(\beta^1, \ldots, \beta^r)$. Using 3.19, both items reduce to already established properties. □

Definition 4.13. An element $\beta = \mathrm{succ}_{i \in \mathrm{In}_\beta} \beta_i \in \mathbf{ord}$ is said to be *filtering* if for each $F \subseteq_f \mathrm{In}_\beta$ there exists $j \in \mathrm{In}_\beta$ such that $\beta_j \geqslant \sup_{i \in F} \beta_i$.

Lemma 4.14. *For each* $\alpha \in$ **ord**, *there exists a* $\beta \in$ **ord** *such that* $\alpha =_{\mathbf{Ord}} \beta$ *and* β *is filtering.*

Proof. If $\alpha = \mathrm{succ}_{i \in J} \alpha_i$, we let K be the set of finitely enumerated subsets of J, and for $F \subseteq_f J$ we let $\beta_F = \sup_{j \in F} \alpha_j$. Then $\beta = \mathrm{succ}_{F \in K} \beta_F$. □

4.3 Elementary ordinal arithmetic

(Sequential) addition

The "sequential" addition $\alpha + \beta$ (α followed by β, addition is not commutative) is defined by induction on β:

$$\alpha + \underline{0} = \alpha \quad \text{and} \quad \alpha + \beta = \mathrm{succ}_{j \in \mathrm{In}_\beta}(\alpha + \beta_j) \ \text{ if } \ \beta = \mathrm{succ}_{j \in \mathrm{In}_\beta} \beta_j \in \mathbf{ord}^\star.$$

This formula works only for the case $\mathrm{In}_\beta \neq \mathbb{N}_0$: it would yield $\alpha + \underline{0} = \underline{0}$ if $\mathrm{In}_\beta = \mathbb{N}_0$.
We also have $\alpha + \beta = \sup_{j \in \mathrm{In}_\beta}(\alpha + \beta_j + \underline{1})$ if $\mathrm{In}_\beta \neq \mathbb{N}_0$.
Some properties that can be proved by induction:

- if $\alpha \leqslant \alpha'$ and $\beta \leqslant \beta'$, then $\alpha + \beta \leqslant \alpha' + \beta'$;
- $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$;

- $\alpha + \underline{0} = \underline{0} + \alpha = \alpha$;
- if $\alpha + \beta \leqslant \alpha + \gamma$, then $\beta \leqslant \gamma$;
- if $\beta < \gamma$, then $\alpha + \beta < \alpha + \gamma$;
- $\alpha = \underline{1} + \alpha$ if and only if $\omega \leqslant \alpha$;
- if $\alpha \leqslant \gamma$, then there is a β such that $\gamma = \alpha + \beta$;
- if $\alpha < \gamma$, then there is a $\beta > \underline{0}$ such that $\gamma = \alpha + \beta$.

Sequential indexed sum

Let $J \in \mathfrak{F}$ and consider a well-founded linear order relation $\prec$ on J with a detachable minimal element 0_J. Let $(\beta^j)_{j \in J}$ be an element of $\mathrm{Fam}(J, \mathbf{Ord})$. The "sequential $\prec$-indexed sum" $\sum_{j \prec \ell} \beta^j$ is defined by induction on $\ell \in (J, \prec)$:

$$\sum\nolimits_{j \prec 0_J} \beta^j = 0_J \quad \text{and} \quad \sum\nolimits_{j \prec \ell} \beta^j = \sup_{k \prec \ell} \Big(\big(\sum\nolimits_{j \prec k} \beta^j \big) + \beta^k \Big) \ \text{ if } 0_J \prec \ell.$$

We show by induction on $\prec$ that, given two families $(\beta^j)_{j \in J}$ and $(\gamma^j)_{j \in J}$ such that $\beta^j \leqslant \gamma^j$ for all $j \in J$, we have $\sum_{j \prec \ell} \beta^j \leqslant \sum_{j \prec \ell} \gamma^j$ for all $\ell \in J$.

Remark 4.15. This construction allows us to define a map $\mathbf{ord}_2^{\mathrm{Br}} \to \mathbf{ord}_2$, where $\mathbf{ord}_2^{\mathrm{Br}}$ is the set of names of Brouwer ordinals. See Troelstra [1969] and Brouwer [1918, 1926]. Troelstra only treats countable Brouwer ordinals.

Multiplication

We define $\alpha \cdot \beta$ by induction on $\beta \in \mathbf{ord}$:

$$\alpha \cdot \underline{0} = \underline{0} \quad \text{and} \quad \alpha \cdot \beta = \sup_{j \in \mathrm{In}_\beta} (\alpha \cdot \beta_j + \alpha) \ \text{ if } \ \beta = \mathrm{succ}_{j \in \mathrm{In}_\beta} \beta_j \in \mathbf{ord}^\star.$$

Some properties that can be proved by induction:

- if $\alpha \leqslant \alpha'$ and $\beta \leqslant \beta'$, then $\alpha \cdot \beta \leqslant \alpha' \cdot \beta'$;
- $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$;
- $\alpha \cdot \underline{1} = \underline{1} \cdot \alpha = \alpha$;
- $\alpha \cdot (\beta + \gamma) = (\alpha \cdot \beta) + (\alpha \cdot \gamma)$;
- if $\underline{1} \leqslant \alpha$ and $\alpha \cdot \beta \leqslant \alpha \cdot \gamma$, then $\beta \leqslant \gamma$;
- if $\underline{1} \leqslant \alpha$ and $\beta < \gamma$, then $\alpha \cdot \beta < \alpha \cdot \gamma$.

Exponentiation

We define α^β by induction on $\beta \in \mathbf{ord}$:

$$\alpha^{\underline{0}} = \underline{1} \quad \text{and} \quad \alpha^\beta = \sup\nolimits_{j \in \mathrm{In}_\beta} (\alpha^{\beta_j} \cdot \alpha) \ \text{ if } \ \beta = \mathrm{succ}_{j \in \mathrm{In}_\beta} \beta_j \in \mathbf{ord}^\star.$$

Ackermann

It is possible to continue this "elementary arithmetic" *à la manière de* Ackermann. We define by induction an ordinal $\underline{\mathrm{Ack}}(\alpha,\beta,\gamma)$ that we get "by iterating γ times the preceding map, initialised at α", i.e., more precisely,

$$\begin{array}{rcl}\underline{\mathrm{Ack}}(\alpha,\beta,0) & = & \alpha+\beta \\ \underline{\mathrm{Ack}}(\alpha,0,\gamma) & = & \alpha \qquad \text{if } \ \gamma\in\mathbf{ord}^\star \\ \underline{\mathrm{Ack}}(\alpha,\beta,\gamma) & = & \sup_{k\in\mathrm{In}_\gamma}\sup_{j\in\mathrm{In}_\beta}\underline{\mathrm{Ack}}(\alpha,\underline{\mathrm{Ack}}(\alpha,\beta_j,\gamma),\gamma_k) \\ & & \qquad\qquad \text{if } \ \beta=\mathrm{succ}_{j\in\mathrm{In}_\beta}\beta_j \ \text{ and } \ \gamma=\mathrm{succ}_{k\in\mathrm{In}_\gamma}\gamma_k.\end{array}$$

In particular, $\varepsilon_0=\underline{\mathrm{Ack}}(\omega,\omega,\underline{1})$.

4.4 Fixed-point theorems, big ordinals

5 Countable ordinals

5.1 First steps

As previously indicated, we get countable ordinals when we choose as indexor set

$$\mathfrak{F}_2=\{\,\mathbb{N}_k\,|\,k\in\mathbb{N},k>0\,\}\cup\{\mathbb{N}\}\,,$$

with convenient operations for disjoint unions. We note $\mathbf{ord}_2$ and $\mathbf{Ord}_2$ for $\mathbf{ord}_{\mathfrak{F}_2}$ and $\mathbf{Ord}_{\mathfrak{F}_2}$.

Thus $\mathbf{Ord}_2$ is the set of ordinals of the second class and $\mathbf{ord}_2$ is a set of names for elements of $\mathbf{Ord}_2$.

Lemma 5.1. *Any countable ordinal is the succ of a nondecreasing sequence of countable ordinals.*

Proof. This is Lemma 4.14. □

Proposition 5.2. *Assume* ***LPO****. Then for* $\alpha,\beta\in\mathbf{Ord}$*, we have* $\alpha\leqslant\beta$ *or* $\beta<\alpha$*.*

Proof. Same proof as for Proposition 3.10, in the countable case. □

5.2 Comparison with Martin-Löf ordinals

The set $\mathbf{Ord}_2^{\mathrm{ML}}$ of Martin-Löf ordinals is based on the set $\mathbf{ord}_2$, but the equivalence relation for equality is coarser in $\mathbf{Ord}_2^{\mathrm{ML}}$ than in $\mathbf{Ord}_2$.

Example 3.21 gives ordinals ε and $\varepsilon'\in\mathbf{ord}_2$ for which **LPO** implies "$\mathrm{succ}(\varepsilon)=_{\mathbf{Ord}_2}\varepsilon'$". Thus $\mathrm{succ}(\varepsilon)=\varepsilon'$ in $\mathbf{Ord}_2^{\mathrm{ML}}$. But $\mathrm{succ}(\varepsilon)=_{\mathbf{Ord}_2}\varepsilon'$ implies **LPO**.

5.3 Ordinals as trees

Martin-Löf proposes to visualise an ordinal as a well-founded tree with finite or countable branchings. Such a tree can be defined as the set of its nodes, or branching points, suitably named. In the sequel, In_α is an element of $\mathfrak{F}_2$.

- The tree with only its root represents $\underline{0}$.
- If $(t_i)_{i\in\mathrm{In}_\alpha}$ is a family of ordinal trees for a family of ordinals $(\alpha_i)_{i\in\mathrm{In}_\alpha}$, the upper bound $\alpha=\mathrm{succ}_{i\in\mathrm{In}_\alpha}\alpha_i$ of the α_i+1's is given by the ordinal tree for which there are $\#\mathrm{In}_\alpha$ branches above the root: the branch indexed by $i\in\mathrm{In}_\alpha$ is a copy of t_i.

More formally, we may consider the set $\mathrm{Lst}(\mathbb{N})$ of finite lists of elements of $\mathbb{N}$. Let $n \in \mathbb{N}$ and $\ell, \ell' \in \mathrm{Lst}(\mathbb{N})$. We denote by $n \frown \ell$ the list $[n, \ell_1, \ldots, \ell_k]$, where $\ell = [\ell_1, \ldots, \ell_k]$. We also denote by $\ell \frown n$ the list $[\ell_1, \ldots, \ell_k, n]$, and by $\ell \frown \ell'$ the concatenation of the lists ℓ and ℓ'.

We remark that $\mathrm{Lst}(\mathbb{N})$ can be enumerated in a natural way[4] and that the notion of an $\mathbb{N}$-indexed family in $\mathrm{Lst}(\mathbb{N})$ corresponds, via such an enumeration, to the basic (undefined) notion of map from $\mathbb{N}$ to $\mathbb{N}$.

We define a *well-founded tree with finite or countable branchings* as a detachable subset of $\mathrm{Lst}(\mathbb{N})$ satisfying the two following properties:

- T is stable by "initial segment": if $\ell \in \mathrm{Lst}(\mathbb{N})$, $p \in \mathbb{N}$ and $\ell \frown p \in T$ then $\ell \in T$;
- T contains no infinite branch: more precisely, the initial segments of an infinite list are outside of the tree from a certain step on.

An *ordinal tree* is thus a well-founded tree with finite or countable branchings which is inductively constructed according to the previously indicated process. Thus, to each $\alpha \in \mathbf{ord}_2$, we associate a tree, defined as a suitable subset of $\mathrm{Lst}(\mathbb{N})$, noted $\mathrm{Tree}(\alpha)$. We get in this way a bijection between $\mathbf{ord}_2$ and the set of ordinal trees which are certain well-founded trees with finite or countable branchings.

E.g., if $n \in \mathbb{N}$, the ordinal $\underline{n}$ can be represented by the tree with unary branchings of $n+1$ nodes, described by the following finite sequence of $n+1$ lists $[\,]$, $[0]$, $[0,0]$, $\ldots$, $[0,\ldots,0]$.

The first infinite ordinal ω can be represented by the subset of $\mathrm{Lst}(\mathbb{N})$ enumerated by the infinite sequence $[\,]$, $[0]$, $[1]$, $[1,0]$, $[2]$, $[2,0]$, $[2,0,0]$, $[3]$, $[3,0]$, $[3,0,0]$, $[3,0,0,0]$, etc.

The ordinal $\omega+1$ can be represented by the infinite sequence $[\,]$, $[0]$, $[0,1]$, $[0,1,0]$, $[0,2]$, $[0,2,0]$, $[0,2,0,0]$, $[0,3]$, $[0,3,0]$, $[0,3,0,0]$, $[0,3,0,0,0]$, etc.

The ordinal $\omega+2$ can be represented by the infinite sequence $[\,]$, $[0]$, $[0,0]$, $[0,0,1]$, $[0,0,1,0]$, $[0,0,2]$, $[0,0,2,0]$, $[0,0,2,0,0]$, $[0,0,3]$, $[0,0,3,0]$, $[0,0,3,0,0]$, $[0,0,3,0,0,0]$, etc.

The ordinal $\omega+\omega$ can be represented by the infinite "double sequence" $[\,]$, $[0]$, $[0,1]$, $[0,1,0]$, $[0,2]$, $[0,2,0]$, $[0,2,0,0]$, $[0,3]$, $[0,3,0]$, $[0,3,0,0]$, $[0,3,0,0,0]$, $\ldots$, $[1]$, $[1,0]$, $[1,0,1]$, $[1,0,1,0]$, $[1,0,2]$, $[1,0,2,0]$, $[1,0,2,0,0]$, $[1,0,3]$, $[1,0,3,0]$, $[1,0,3,0,0]$, $[1,0,3,0,0,0]$, $\ldots$, $[2]$, $[2,0]$, $[2,0,0]$, $[2,0,0,1]$, $[2,0,0,1,0]$, $[2,0,0,2]$, $[2,0,0,2,0]$, $[2,0,0,2,0,0]$, $[2,0,0,3]$, $[2,0,0,3,0]$, $[2,0,0,3,0,0]$, $[2,0,0,3,0,0,0]$, etc.

`DESSINER QUELQUES ARBRES`

5.4 Computable ordinals à la Kleene

6 A geometric theory for ordinals in classical mathematics

In classical mathematics we can replace the nongeometric axioms (Axioms 12, 13 and 14) of the theory of $\mathfrak{F}$-orders with the following (finitary or infinitary) dynamic rules.

`Axsucc0.` For each $(\alpha_i)_{i\in I}$ in $\mathrm{Fam}(\mathfrak{F}, E)$ and each $i \in I$, we have $\vdash\ \alpha_i < \alpha$ (where $\alpha = \mathrm{succ}_{i\in I}\, \alpha_i$).

`Axsucc` For each $(\alpha_i)_{i\in I}$ in $\mathrm{Fam}(\mathfrak{F}, E)$ and each $\beta \in E$, we have $\vdash\ \alpha \leqslant \beta \ \vee\ \bigvee_{i\in I} \beta \leqslant \alpha_i$ (where $\alpha = \mathrm{succ}_{i\in I}\, \alpha_i$).

`Axsup0.` For each $(\alpha_i)_{i\in I}$ in $\mathrm{Fam}(\mathfrak{F}, E^\star)$ and each $i \in I$, we have $\vdash\ \alpha_i \leqslant \gamma$ (where $\gamma = \sup_{i\in I} \alpha_i$).

`Axsup.` For each $(\alpha_i)_{i\in I}$ in $\mathrm{Fam}(\mathfrak{F}, E^\star)$ and each $\beta \in E$, we have $\vdash\ \gamma \leqslant \beta \ \vee\ \bigvee_{i\in I} \beta < \alpha_i$ (where $\gamma = \sup_{i\in I} \alpha_i$).

$\mathtt{Ax}_{<\leqslant}$. For all $\alpha, \beta \in E$ we have $\vdash\ \alpha \leqslant \beta \ \vee\ \bigvee_{\gamma:\gamma<\alpha} \beta \leqslant \gamma$.

[4] For example, for $\ell = [\ell_1, \ldots, \ell_k] \in \mathrm{Lst}(\mathbb{N})$, we let $\mu(\ell) = \sum_{i=1}^{k} (1+\ell_i)$ and we enumerate the lists by increasing $\mu(\ell)$.

One sees that $\mathtt{Ax}_{<\leqslant}$ follows directly from `Axsucc`.

We get in this way a geometric theory $\mathcal{T}_{\mathfrak{F}}$. We can analyse constructively this theory even if it has no known constructive model.

We can endow $\mathbf{ord}_2$ with the equivalence relation corresponding to provable equality of objects in $\mathcal{T}_{\mathfrak{F}_2}$. This defines the quotient $\mathbf{Ord}_2^{\mathrm{cl}}$, which we see intuitively as the set of ordinals in classical mathematics viewed in constructive mathematics.

A first question would be: "Is $\mathbf{Ord}_2^{\mathrm{cl}} = \mathbf{Ord}_2^{\mathrm{ML}}$?" If not, we need an example of two elements of $\mathbf{ord}_2$ that are equal in $\mathbf{Ord}_2^{\mathrm{cl}}$ but not in $\mathbf{Ord}_2^{\mathrm{ML}}$.

Do Martin-Löf's ordinals satisfy more or less $\mathtt{Ax}_{<\leqslant}$ or `Axsucc`? This has to be made precise.

Perhaps there are certain conservativity properties of $\mathcal{T}_{\mathfrak{F}_2}$ with respect to $\mathbf{Ord}_2$.

This could be an interesting topic: a way of indicating that **LPO** is without danger in certain contexts.

7 Beyond countable ordinals

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