LECTURE 1. SMOOTH MORPHISMS.

DORIN POPESCU

A ring morphism $h : A \to B$ is called *quasi-smooth* if for any A-algebra D and an ideal $I \subset D$ with $I^2 = 0$, any A-morphism $B \to D/I$ lifts to an A-morphism $B \to D$. If any such lifting is unique, then h is *quasi-etale*. A quasi-smooth (resp. *quasi-etale*) is called *smooth* (resp. *etale*) if it is finitely presented and it is called *essentially smooth* (resp. *essentially etale*) if it is a localization of a smooth (resp. etale) morphism.

Example 1. i) A polynomial algebra $A[X_1, \dots, X_n]$ is smooth over A.

ii) A localization A_S is essentially etale over A.

iii) If $f = (f_1, \dots, f_n) \in A[X_1, \dots, X_n]^n$ with $\Delta = \det(\partial f_i/\partial X_j) \neq 0$ then $B = (A[X_1, \dots, X_n]/(f))_\Delta$ is etale. Indeed, a morphism $g : B \to D/I$ is given by $X_i \to (x_i + I)$ for some $x_i \in D$. To lift g to a morphism $B \to D$ means to find $t_i \in I$ such that $f(x_1 + t_1, \dots, x_n + t_n) = 0$. By Taylor's formula this is equivalent to find a solution to the linear system of equations $\sum_i (\partial f_i/\partial X_j)(x_1, \dots, x_n)T_j + f_i(x_1, \dots, x_n) = 0$, $i \in [n]$. This holds uniquely applying Cramer's rule, which means that B/A is etale.

iv) Now let $f = (f_1, \dots, f_r) \in A[X_1, \dots, X_n]^r$, $r \leq n$ and Δ be some $r \times r$-minor of the Jacobian matrix $(\partial f_i/\partial X_j)$. Then a similar proof shows that $B = (A[X_1, \dots, X_n]/(f))_\Delta$ is smooth over A. In particular, if $r = 1$ and $f' = \partial f/\partial X_n$ then $B = (A[X_1, \dots, X_n]/(f))_{f'}$ is smooth over A.

Remark 2. Conversely, Grothendieck showed in [2] (see also [7, Theorem 2.5]) that any smooth A-algebra looks locally like $(A[X_1, \dots, X_n]/(f))_{f'}$ as above.

Lemma 3. *Let B be a quasi-smooth (resp. quasi-etale) A-algebra and $g : B \to C$ a morphism of A-algebras such that C is quasi-smooth (resp. quasi-etale) over B, the structure of B-algebra of C being given by g. Then C is quasi-smooth (resp quasi-etale) over A.*

Proof. Let D be an A-algebra, $J \subset D$ an ideal and $\bar{w} : C \to D/J$ an A-morphism. Then there exists an A-morphism $v : B \to D$ lifting $\bar{w}g$ because B is quasi-smooth over A. Thus D has a structure of B-algebra via v and there exists a B-algebra lifting $w : C \to D$ of $\bar{w}$ because C is quasi-smooth over B. Clearly, w is also an A-algebra lifting of $\bar{w}$. The quasi-etale case goes similarly. $\square$

Lemma 4. *(Base change) Let $u : A \to B$ be a quasi smooth morphism and C an A-algebra. Then $C \otimes u$ is also quasi smooth.*

Let $B = A[X_1, \dots, X_n]/I$, F the free B-module with basis dX_i, $i \in [n]$ and $d : I \to F$ the map defined for $f \in I$ by $d(f) = \sum_{i=1}^n (\partial f/\partial X_i)dX_i$. As $d(I^2) = 0$ note that d induces a map $d_I : I/I^2 \to F$.

Proposition 5. *B is quasi-smooth if and only if d_I has a retraction.*

Proof. Let D be an A-algebra, $J \subset D$ an ideal with $J^2 = 0$ and $g : B \to D/J$ an A-morphism, let us say g is given by $X_i \to x_i + J$, $i \in [n]$ for some $x_i \in D$. Let $h : A[X_1, \dots, X_n] \to D$ be the map given by $f \to f(x)$. We have $h(I) \subset J$ and so h induces a map $h_I : I/I^2 \to J$ because $J^2 = 0$.

As in Example 1 we note that g can be lifted to a map $B \to D$ if there exists $t = (t_1, \dots, t_n)$ in J which is a solution of the linear system of equations

$$\sum_{j=1}^{n} (\partial f/\partial X_j)(x) T_j + f(x) = 0$$

for all $f \in I$. Since $J^2 = 0$ we see that J is a D/J-module and so a B-module via g. Therefore, g can be lifted to a map $B \to D$ if and only if there exists a B-module morphism $\varphi : F \to J$ (in our case given by $dX_j \to t_j$) such that $\varphi d_I = -h_I$.

Now the sufficiency is trivial because if d_I has a retraction ψ then we may take $\varphi = -h\psi$. Conversely, take $D = A[X_1, \dots, X_n]/I^2$, $J = I/I^2$, $g = 1_B$ and $h_I = 1_{I/I^2}$. Then there exists a lifting $B \to D$ and so as above there exists $\varphi : F \to J$ such that $\varphi d_I = -h_I$. Thus d_I has a retraction. □

Lemma 6. *B is smooth over A if and only if B_m is smooth over A for every maximal ideal m of B.*

Proof. Let $d_I : I/I^2 \to F$ be as above. Using the above proposition it is enough to see that d_I has a retraction if and only if $B_m \otimes d_I$ has a retraction for each maximal ideal m of B as shows the following elementary lemma. □

Lemma 7. *Let $v : M \to N$ be an A-morphism of finitely presented A-modules. Then v has a retraction if and only if $A_m \otimes v$ has a retraction for all maximal ideal m of A.*

Proof. Let $v' : \mathrm{Hom}_A(N, M) \to \mathrm{Hom}_A(M, M)$ be the map given by $g \to gv$. Then v has a retraction if and only if $1_M \in \mathrm{Im}\, v'$, that is if and only if v' is surjective. Note that $(\mathrm{Hom}_A(N, M))_m \cong \mathrm{Hom}_{A_m}(N_m, M_m)$ because N, M are finitely presented A-modules. Clearly, v' is surjective if and only if $A_m \otimes v'$ is surjective for all maximal ideal m of A, which is enough. □

A ring morphism $u : A \to A'$ has *regular fibers* if for all prime ideal $Q \in \mathrm{Spec}\, A'$ the ring $(A'/u^{-1}(Q)A')_Q$ is a regular local ring. The purpose of our lectures will be to sketch a proof to the following theorem.

Theorem 8. *(General Neron Desingularization, Popescu* [3], [4], [5]*) Suppose that $A \supset \mathbf{Q}$ and let $u : A \to A'$ be a regular morphism of noetherian rings. Then any A-morphism $v : B \to A'$ factors through a smooth A-algebra C, that is v is a composite A-morphism $B \to C \to A'$.*

Let $B = A[X_1, \dots, X_n]/(f)$ for some systems of polynomials f. Roughly speaking the above theorem says that one can extend the solution $v(X)$ of f to a solution of a bigger system of polynomial equations g from $A[X_1, \dots, X_n, Y_1, \dots, Y_N]$ such that $C = A[X_1, \dots, X_n, Y_1, \dots, Y_N]/(g)$ is a smooth A-algebra.

Remark 9. The above theorem is also true when A does not contain $\mathbf{Q}$ but then we must suppose that u is the so called geometrically regular. Also the conclusion of this theorem could say that A' is a filtered colimit of smooth A-algebras. Clearly, the last statement implies the previous one. The converse is also easy (see [1, Lemma 5.2], or [7, Lemma 1.5]) but probably we will not use in our lectures.

Example 10. Let K/k be a separable field extension, g a system of polynomial equations from $k[Y]$, $Y = (Y_1, \dots, Y_n)$ and $y \in K^n$ a solution of g in K. Then there exists a finite type field subextension E/k of K/k such that $y \in E^n$. Since E/k is separable and of finite type, it is separable generated, let us say $E = k(x_1, \dots, x_r)$, where $x_1, \dots, x_{r-1}$ is a transcendental basis over k and x_r is a primitive algebraic separable element over $k(x_1, \dots, x_{r-1})$. Let $f = Irr(k(x_1, \dots, x_{r-1}), x_r) \in k(x_1, \dots, x_{r-1})[X_r]$. Then the partial derivation $f' = \partial f/\partial X_r$ satisfies $f'(x_r) \neq 0$ and $E = Q(k[X]/(f))$ is essentially smooth over k. Clearly the map $v : B = k[Y]/(g) \to K$ factors through a smooth k-algebra C of type $(k[X]/(f))_{f'h}$ for some $h \in k[X] \setminus (f)$, where we identify X_i with x_i for $i < r$.

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LECTURE 2. THE SMOOTH LOCUS OF A MORPHISM.

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Let B be finitely presented A-algebra, let us say $B = A[X_1, \ldots, X_n]/I$, where I is finitely generated. If $f = (f_1, \ldots, f_r)$, $r \leq n$ is a system of polynomials from I then we can define the ideal Δ_f generated by all $r \times r$-minors of the Jacobian matrix $(\partial f_i/\partial X_j)$. After Elkik [2] let $H_{B/A}$ be the radical of the ideal $\sum_f ((f) : I)\Delta_f B$, where the sum is taken over all systems of polynomials f from I with $r \leq n$. An element $d \in H_{B/A}$ is called *standard* if there exists a system of r-polynomials f from I, $r \leq n$ such that $d \in \sqrt{((f) : I)\Delta_f B}$. If $d \in ((f) : I)\Delta_f B$ for some f then we call *a strictly standard.*

Remark 1. Let $S \subset B$ be a multiplicative set in B. Then $(H_{B/A})_S = H_{B_S/A}$.

We would like to prove a stronger form as I state in the first lecture.

Theorem 2. *(General Neron Desingularization, Popescu* [5], [6], [7]*) Suppose that $A \supset \mathbf{Q}$ and let $u : A \to A'$ be a regular morphism of noetherian rings. Then any A-morphism $v : B \to A'$ can be factorized through a standard smooth A-algebra B', that is v is a composite A-morphism $B \to B' \to A'$.*

Lemma 3. *Let K/k be a separable field extension, (A, m) an Artinian local k-algebra, $\tilde{A} = A \otimes_A K$ and $A' = (\tilde{A})_{m\tilde{A}}$. Then A' is an Artinian local K-algebra with the maximal ideal mA' and any A-morphism $v : B \to A'$ can be factorized through a standard smooth A-algebra B', that is v is a composite A-morphism $B \to B' \to A'$.*

Proof. As in the last example of the first lecture we express $K = \cup_i k_i$ as a filtered union of finite type field extensions. Set $A_i = A \otimes_k k_i$. Then $A = \cup_i A_i$ is a filtered union of local k-subalgebras of A'. We have $k_i = Q(k[X_1, \ldots, X_r]/(g))$ for some monic polynomial g with $g' = \partial g/\partial X \notin (g)$ and so $A_i \cong (A[X_1, \ldots, X_r]/(g))_{(m,g)}$. Thus A' is a filtered union of essential smooth A-algebras A_i. Since B is of finite type over A, v factors through A_i for some i. But A_i is a filtered inductive limit of standard smooth algebras of type $B' = A[X_1, \ldots, X_r]/(g))_{g'h}$ for some $h \in A[X_1, \ldots, X_r] \setminus (m, g)$ and so v factors through such B'. □

Theorem 4. *General Neron Desingularization holds for Artinian local rings.*

Proof. Let (A, m, k) be a local Artinian ring containing $\mathbf{Q}$ and $u : A \to A'$ be a regular morphism of Artinian local rings. Then mA' is the maximal ideal of A' and let $k' = A'/mA'$. By [3] there exists in general (that is even A is not Artinian but Noetherian) a flat complete Noetherian local A-algebra C, unique up to an isomorphism, such that mC is the maximal ideal of C and $C/mC \cong k'$. As any Artinian local ring is complete we see that in fact A' is unique given with the above properties. But in the above lemma we constructed one A-algebra A' of this type as a localization of $A \otimes_k k'$. By unicity this should be our A' and it is enough to apply the above lemma. $\square$

Next we will give an application of Theorem 2 which answers a Conjecture of M. Artin [1].

Theorem 5. *(Popescu [6], [8]) An excellent Henselian local ring has the property of approximation.*

Proof. Let (A, m) be an excellent Henselian local ring, h be a finite system of polynomial equations in $Z = (Z_1, \ldots, Z_s)$ over A and $\tilde{z}$ a solution of h in the completion $\hat{A}$ of A. By General Neron Desingularization the A-morphism $v : B = A[Z]/(h) \to \hat{A}$, $Z \to \tilde{z}$ factors through an A-algebra $C = (A[Y]/(f))_g$ $Y = (Y_1, \ldots, Y_N)$, where $f = (f_1, \ldots, f_r)$, $r \leq N$, are polynomials in Y over A and g belongs to the ideal Δ_f generated by all $r \times r$-minors of the Jacobian matrix $(\partial f_i/\partial Y_j)$, let us say $v = wq$, $w : C \to \hat{A}$, $q : B \to C$. Then $\hat{y} = w(\hat{Y})$ is a solution of f in $\hat{A}$ such that $g(\hat{y}) = w(\hat{g}) \notin m\hat{A}$. Choose $\tilde{y}$ in A^n such that $\tilde{y} \equiv \hat{y} \bmod m\hat{A}$. Then $f(\tilde{y}) \equiv f(\hat{y}) = 0 \bmod m\hat{A}$, $g(\tilde{y}) \equiv g(\hat{y}) \neq 0 \bmod m\hat{A}$. In particular, $f(\tilde{y}) \equiv 0 \mod m$. By the Implicit Function Theorem we get a solution y of f in A such that $y \equiv \tilde{y} \mod m$. Then we get an A-morphism $u : C \to A$ by $Y \to y$. Clearly, $z = uq(\hat{Z})$ is a solution of h in A such that $z \equiv \hat{z} \mod m\hat{A}$. $\square$

Lemma 6. *Let A be a ring and a_1, a_2 a weak regular sequence of A, that is a_1 is a non-zero divisor of A and a_2 is a non-zero divisor of $A/(a_1)$. Let A' be a flat A-algebra and set $B = A[X_1, X_2]/(f)$, where $f = a_1X_1 + a_2X_2$. Then $H_{B/A}$ is the radical of (a_1, a_2) and any A-morphism $B \to A'$ factors through a polynomial A-algebra in one variable.*

Proof. Note that all solutions of $f = 0$ in A are multiples of $(-a_2, a_1)$. By [4, Theorem 7.6]) any solution of f in A' is a linear combinations of some solutions of f in A and so again a multiple of $(-a_2, a_1)$.

Now let $h : B \to A'$ be a map given by $X_i \to x_i \in A'$. Then $(x_1, x_2) = z(-a_2, a_1)$ and so h factors through $A[Z]$, that is h is the composite map $B \to A[Z] \to A'$, first map being given by $(X_1, X_2) \to Z(-a_2, a_1)$ and the second one by $Z \to z$. $\square$

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LECTURE 3. ELKIK'S THEOREM.

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In the idea of the last lemma from Lecture 2 we present the following proposition.

Proposition 1. *Let $f_i = \sum_{i=1}^n a_{ij} X_j \in A[X_1, \ldots, X_n]$, $i \in [N]$ be a system of linear homogeneous polynomials and $q^{(k)} = (q_1^{(k)}, \ldots, q_n^{(k)})$, $k \in [p]$ be a complete system of solutions of $f = (f_1, \ldots, f_N) = 0$ in A. Let $b = (b_1, \ldots, b_N) \in A^N$ and c a solution of $f = b$ in A. Let A' be a flat A-algebra and $B = A[X_1, \ldots, X_n]/(f - b)$. Then any A-morphism $B \to A'$ factors through a polynomial A-algebra in p variables.*

Proof. Let $h : B \to A'$ be a map given by $X \to x \in A'^n$. Since A' is flat over A we see that $x - h(c)$ is a linear combinations of $q^{(k)}$, that is there exist $z = (z_1, \ldots, z_p) \in A'^p$ such that $x - h(c) = \sum_{k=1}^p z_k h(q^{(k)})$. Therefore, h factors through $A[Z_1, \ldots, Z_p]$, that is h is the composite A-morphism $B \to A[Z_1, \ldots, Z_p] \to A'$, where the first map is given by $X \to c + \sum_{k=1}^p Z_k q^{(k)}$ and the second one by $Z \to z$. $\square$

Theorem 2. *(Elkik [1]) Let B be a finitely presented A-algebra and P a prime ideal of B. Then B_P is essentially smooth over A if and only if $P \not\supset H_{B/A}$.*

Proof. Let $B = A[X_1, \ldots, X_n]/I$. Suppose that $P \not\supset H_{B/A}$ and let $Q \subset A[X_1, \ldots, X_n]$ be the inverse image of P. Let f be a system of r polynomials from I such that $((f) : I)\Delta_f \not\subset P$. Then $\mathrm{Ann}_{A[X_1,\ldots,X_n]} I/(f) \not\subset Q$ and so f generates I_Q. We may suppose that $\det(\partial f_i/\partial X_j)_{i,j=1,\ldots,r} \notin Q$ because $\Delta_f \not\subset P$. Then the composite map

$$(B_P)^r \to (I/I^2)_P \to \sum_{i=1}^n B_P dX_i \to \sum_{i=1}^r B_P dX_i$$

given by $e_i \to f_i \to \sum_{j=1}^n (\partial f_i/\partial X_j) dX_j \to \sum_{j=1}^r (\partial f_i/\partial X_j) dX_j$ is invertible because $\det(\partial f_i/\partial X_j)_{i,j=1,\ldots,r} \notin Q$, (e_i) being the canonical basis of $(B_P)^r$. The first map is surjective since f generates I_Q and it follows also injective from above. Therefore $B_P \otimes_B d_I$ has a retraction and so B_P is quasi-smooth over A.

Conversely, we may suppose that $B_P \otimes_B d_I$ has a retraction. It follows that $(I/I^2)_P$ is projective and so free over the local ring B_P. Choose f to be a system of let us say r polynomials from I inducing a basis in $(I/I^2)_P$. By Nakayama we see that f generates I_Q. Since $(B_P)^r \to (I/I^2)_P \to \sum_{i=1}^n B_P dX_i$ has a retraction we see that 1 is a linear combinations of $r \times r$-minors of $(\partial f/\partial X)$. Thus we may assume that $M = \det(\partial f_i/\partial X_j)_{i,j=1,\ldots,r} \notin Q$ and so $\Delta_f \not\subset Q$. Therefore, $H_{B/A} \not\subset P$. $\square$

Remark 3. The above theorem says that $V(H_{B/A})$ is the non smooth locus of B over A.

Corollary 4. *B is smooth if and only if $H_{B/A} = B$.*

Proof. B is smooth over A if and only if B_m is essentially smooth over A for every maximal ideal m of B. Using the above theorem we see that the last statement says that there exists no maximal ideal of B containing $H_{B/A}$, that is $H_{B/A} = B$. $\square$

Corollary 5. *Let $s \in B$. Then B_s is smooth over A if and only if $s \in H_{B/A}$.*

Proof. Note that B_s is smooth over A if and only if $(H_{B/A})_s = H_{B_s/A} = B_s$ by the above corollary. Thus $s \in H_{B/A}$. $\square$

Corollary 6. *Let $g : B \to C$ be a morphism of finitely presented A-algebras. Then $H_{C/A} \supset g(H_{B/A})C \cap H_{C/B}$, the structure of B-algebra of C being given by g.*

Proof. Indeed, let $P \in \operatorname{Spec} C$, $P \not\supset g(H_{B/A})C \cap H_{C/B}$. Then $g^{-1}(P) \not\supset H_{B/A}$ and $P \not\supset H_{C/B}$ and so $B_{g^{-1}(P)}$ is essentially smooth over A and C_P is essentially smooth over B by Theorem 2. Thus C_P is essentially smooth over A, that is $P \not\supset H_{C/A}$ using again Theorem 2. $\square$

Lemma 7. *Let $u : A \to A'$ be a morphism, B be a finitely presented A-algebra, $y \in B$ and $v : B \to A'$ an A-morphism such that $v(y) \in \sqrt{v(H_{B/A})A'}$ but $y \notin H_{B/A}$. Then there exist a finite presentation A-algebra C and two A-morphisms $g : B \to C$, $w : C \to A'$ such that*

(1) *$g(y) \in H_{C/A}$,*
(2) *$v = wg$,*
(3) *$g(H_{B/A}) \subset H_{C/B}$ and so $g(H_{B/A}) \subset H_{C/A}$ by Corollary 6.*

Proof. Let $x = (x_1, \ldots, x_n)$ be a system of generators of $H_{B/A}$. By hypothesis we have $v(y)^s = \sum_{i=1}^n v(x_i)z_i$ for some $s \in \mathbf{N}$ and some elements $z_i \in A'$. Set $C = B[Z_1, \ldots, Z_n]/(y^s - \sum_{i=1}^n x_i Z_i)$ and let $w : C \to A'$ be given by $Z \to z$. We have $x_i \in H_{C/B}$ and so (3) holds. Moreover $g(y)^s = \sum_{i=1}^n v(x_i)Z_i \in H_{C/B}$, that is (1) holds too. $\square$

Lemma 8. *Let $u : A \to A'$ be a morphism, B be a finitely presented A-algebra and $v : B \to A'$ an A-morphism such that $v(H_{B/A}) = A'$. Then v factors through a smooth A-algebra C, that is v is a composite A-morphism $B \to C \to A'$.*

For the proof apply the above lemma for $y = 1$.

Next we will see how goes the proof of General Neron Desingularization on dimension 1. Let $u : A \to A'$ be a flat morphism of Noetherian local rings of dimension 1. Suppose that $A, A' \supset \mathbf{Q}$ are domains and the maximal ideal m of A generates the maximal ideal of A'. Then u is a regular morphism.

Lemma 9. *Let B be a finite type A-algebra and $v : B \to A'$ be an A-morphism. Then v factors through a finite type A-algebra B' such that there exists $d \in A$, $d \neq 0$ strictly standard for B' over A.*

Proof. Clearly we may substitute B by $\operatorname{Im} v$. Suppose that $B = A[X]/I$, $X = (X_1, \ldots, X_n)$. Since $Q(B)/Q(A)$ is a separable field extension we get that $H_{B/A} \neq 0$ and so $h_B \neq 0$ because (0) is a smooth point (the same thing follows applying the Jacobian Criterion). If $h_B = A'$ we may apply Lemma 8. We assume that $h_B = mA'$ and so $h_B \cap A = m$. Let $P \in (\Delta_f((f) : I)) \setminus I$ for some system of polynomials $f = (f_1, \ldots, f_r)$ from I and choose $d' \in (v(P)A') \cap A$. Moreover we may **choose** P to be from $M((f) : I)$ where M is a $r \times r$-minor of $(\partial f/\partial X)$. Then $d' = v(P)z$ for some $z \in A'$. Set $B' = B[Z]/(g)$, $g = d' - PZ$ and let $v' : B' \to A'$ be the map of B-algebras given by $Z \to z$. It follows that $d' \in (((g), f) : (I, g))$ and $d' \in \Delta_f$, $d' \in \Delta_g$. Thus $d = d'^3 \neq 0$ is strictly standard for B' over A. □

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LECTURE 4. GENERAL NERON DESINGULARIZATION IN DIMENSION 1

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The aim of this lecture is to prove General Neron Desingularization in dimension 1. Let $u : A \to A'$ be a flat morphism of Noetherian local rings of dimension 1. Suppose that $A, A' \supset \mathbf{Q}$ are domains and the maximal ideal m of A generates the maximal ideal of A'. Then u is a regular morphism. Let $B = A[X_1, \ldots, X_n]/I$ be a finitely presented A-algebra and $v : B \to A'$ an A-morphism. Using the last lemma from Lecture 3 we may factor v through a finite type A-algebra B' such that there exists $d \in A$, $d \neq 0$ strictly standard for B' over A. Thus we may suppose from the beginning that B has the property of B' changing B by B'. Moreover we may assume that there exists $f = (f_1, \ldots, f_r)$ in I, $M = \det(\partial f_i/\partial X_j)_{i,j\in[r]}$ and $P \in (M((f) : I)) \setminus I$ such that $d \equiv P$ modulo I. Set $\bar{A} = A/(d^3)$, $\bar{A}' = A'/d^3A'$, $\bar{u} = \bar{A} \otimes_A u$, $\bar{B} = B/d^3B$, $\bar{v} = \bar{A} \otimes_A v$. Clearly $\bar{u}$ is a regular morphism of Artinian local rings.

Lemma 1. *(Small Lifting Lemma) Then there exists a standard smooth A-algebra D and an A-morphism $w : D \to A'$ such that $\bar{v}$ factors through $\bar{w} = \bar{A} \otimes_A w$.*

Proof. By General Neron Desingularization in Artinian case, $\bar{v}$ factors through a smooth $\bar{A}$-algebra $\bar{C}$, let us say $\bar{v}$ is the composite map $\bar{B} \to \bar{C} \xrightarrow{\bar{\beta}} \bar{A}'$. Moreover, $\bar{C} = (\bar{A}[Y]/(\bar{g}))_{\bar{g}'\bar{G}}$, $Y = (Y_1, \ldots, Y_r)$, $\bar{g}, \bar{G} \in \bar{A}[Y]$, $\bar{g}' = \partial\bar{g}/\partial Y_r$ and $\bar{g}$ is monic in Y_r.

Let $g, G \in A[Y]$ be polynomials lifting $\bar{g}, \bar{G}$ and suppose that g is monic in Y_r. Choose $y \in A'^r$ inducing $\bar{\beta}(Y)$ in $\bar{A}'$. Then $g(y) = d^3 z$ for some $z \in A'$. Set $D = (A[Y, Z]/(g - d^3 Z))_{g'G}$, $g' = \partial g/\partial Y_r$ and let $w : D \to A'$ be the map $(Y, Z) \to (y, z)$. We call D a *lifting* of $\bar{C}$. Unfortunately, v does not factor through D. But $\bar{v}$ factors through $\bar{D}$ and $\bar{D} = \bar{A} \otimes_A D \cong \bar{C}[Z]$ is smooth over $\bar{C}$ and so over $\bar{A}$. $\square$

Remark 2. *If $A' = \hat{A}$ then $\bar{A} \cong \bar{A}'$ and we may take $D = A$.*

Let $S = B \otimes_A D \cong D[X]/ID[X]$ and $\alpha : S \to A'$ be given by $b \otimes z' \to v(b)w(z')$. Then v, w factor through α. We have the canonical maps $\nu : B \to S$, $b \to b \otimes 1$ and $\gamma : D \to S$, $z' \to 1 \otimes z'$. Unfortunately, S is not smooth over A even it is smooth over B. Since $\bar{v}$ factors through $\bar{D}$ there exists $\bar{\tau} : \bar{B} \to \bar{D}$ such that $\bar{v} = \bar{w}\bar{\tau}$. Then the map $\bar{\rho} : \bar{S} = \bar{A} \otimes_A S \to \bar{D}$ given by $\bar{b} \otimes \bar{z}' \to \bar{\tau}(b)\bar{z}'$ is a retraction of $\bar{D}$-algebras and $\bar{w}\bar{\rho} = \bar{\alpha} := \bar{A} \otimes_A \alpha$. Let $\bar{\rho}$ be given by $X \to x + d^3 D^n$ for some $x \in D^n$. Thus $I(x) \equiv 0$ modulo $d^3 D$

Note that if $A' = \hat{A}$ as in Remark 2 we have $S \cong B$ and the retraction $\bar{\rho}$ is the composite map $\bar{S} \cong \bar{B} \xrightarrow{\bar{v}} \bar{A}' \cong \bar{A} = \bar{D}$.

We have $d \equiv P$ modulo I and so $P(x) \equiv d$ modulo d^3 in D because $I(x) \equiv 0$ modulo d^3D. Thus $P(x) = ds$ for a certain $s \in D$ with $s \equiv 1$ modulo d. Assume that $P = NM$ for some $N \in (I : (f))$. Let H be the $n \times n$-matrix obtained adding down to $(\partial f/\partial X)$ as a border the block $(0|\mathrm{Id}_{n-r})$. Let G' be the adjoint matrix of H and $G = NG'$. We have

$$GH = HG = NM\mathrm{Id}_n = P\mathrm{Id}_n$$

and so

$$ds\mathrm{Id}_n = P(x)\mathrm{Id}_n = G(x)H(x).$$

Let α be given by $X \to y \in A'^n$. Set $x' = w(x)$. We have $y - x' \in d^3A'^n$, let us say $y - x' = d^2\varepsilon$ for $\varepsilon \in dA'^n$. Then $t := H(x)\varepsilon \in dA'^n$ satisfies

$$G(x)t = P(x)\varepsilon = ds\varepsilon$$

and so

$$s(y - x') = dG(x)t.$$

Let

$$h = s(X - x) - dG(x)T,$$

where $T = (T_1, \ldots, T_n)$ are new variables. The kernel of the map $\varphi : D[X, T] \to A'$ given by $X \to y$, $T \to t$ contains h. Since

$$s(X - x) \equiv dG(x)T \text{ modulo } h$$

and

$$f(X) - f(x) \equiv \sum_j \partial f/\partial X_j(x)(X_j - x_j)$$

modulo higher order terms in $X_j - x_j$ by Taylor's formula we see that for $m = \max_i \deg f_i$ we have

$$s^m f(X) - s^m f(x) \equiv \sum_j s^{m-1} d\partial f/\partial X_j(x)G_j(x)T_j + d^2Q =$$

$$s^{m-1}dP(x)T + d^2Q$$

modulo h where $Q \in T^2D[T]^r$. This is because $(\partial f/\partial X)G = (P\mathrm{Id}_r|0)$. We have $f(x) = d^2c$ for some $c \in dD^r$. Then $g_i = s^m c_i + s^m T_i + Q_i$, $i \in [r]$ is in the kernel of φ because $d^2\varphi(g) = d^2g(t) \in (h(y, t), f(y)) = (0)$. Set $E = D[X, T]/(I, g, h)$ and let $\lambda : E \to A'$ be the map induced by φ.

Note that the $r \times r$-minor s' of $(\partial g/\partial T)$ given by the first r-variables T is from $s^{rp} + (T) \subset 1 + (d, T)$ because $Q \in (T)^2$. Then $U = (D[X, T]/(h, g))_{ss'}$ is smooth over D. We claim that $I \subset (h, g)D[X, T]_{ss's''}$ for some other $s'' \in 1 + (d, T)$. Indeed, we have $PI \subset (h, g)D[X, T]_s$ and so $P(x + s^{-1}dG(x)T)I \subset (h, g)D[X, T]_s$. Since $P(x + s^{-1}dG(x)T) \in P(x) + d(T)$ we get $P(x + s^{-1}dG(x)T) = ds''$ for some $s'' \in 1 + (d, T)$. It follows that $s''I \subset (h, g)D[X, T]_{ss'}$ because d is regular in U, the map $A \to U$ being flat, and so $I \subset (h, g)D[X, T]_{ss's''}$. Thus $E_{ss's''} \cong U_{s''}$ is a B-algebra which is also standard smooth over A.

As $w(s) \equiv 1$ modulo d and $w(s'), w(s'') \equiv 1$ modulo (d, t), $d, t \in mA'$ we see that $w(s), w(s'), w(s'')$ are invertible because A' is local and so λ (thus v) factors through the standard smooth A-algebra $E_{ss's''}$.

Remark 3. We believe that the above proof provides a possible algorithm to find the smooth A-algebra $E_{ss'}$.

Remark 4. Assume that A, A' are Noetherian local rings of dimension one but not domains. Then how I can get above $g(t) = 0$ from $d^2 g(t) = 0$? In this case I should replace d by a high power e of it such that $\operatorname{Ann}_A d = \operatorname{Ann}_A d^2$ and to arrange t such that $g(t) \in (d)$ because then $\operatorname{Ann}_A d \cap (d) = 0$. For the first condition we take e such that the chain $\operatorname{Ann}_A d \subset \operatorname{Ann}_A d^2 \subset \ldots \subset \operatorname{Ann}_A d^n \subset \ldots$ stops for $n = e$ by Noetherianity. Could we find an algorithm to compute e?

Remark 5. Suppose that A, A' are domains of dimension 2. Then we might construct D as above with $h_D \not\subset q$ for a minimal prime ideal q containing h_B. But we should also ask to have $h_B \subset h_D$ because otherwise D could mean no progress compare with B. In dimension one we did not need the condition $h_B \subset h_D$ but now this is essential.

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EXTRA LECTURE 1. A VIEW ON THE PROOF OF THE GENERAL NERON DESINGULARIZATION

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This lecture uses mainly pages 147, 168, 169, 171 from [5] and 347-354 of [4].

The standard elements play an important role in the proof of General Neron Desingularization. Let $u : A \to A'$ be a regular morphism of Noetherian rings containing $\mathbf{Q}$, B a finite type A-algebra and $v : B \to A'$ a morphism of A-algebras. We need to reduce whenever it is necessary to the case when all elements of $H_{B/A}$ are standard. This is done by the following theorem.

Theorem 1. *(Elkik [1]) Let $B = A[X_1, \dots, X_n]/I$ be a finitely presented A-algebra, $M = I/I^2$ and $C = S_A(M)$ be the symmetric A-algebra defined by M. Then $H_{B/A}C \subset H_{C/A}$ and any element of $H_{B/A}$ becomes standard in C over A.*

We will not give here the proof, but an easy one is presented on pages 147, 148 of [5].

Corollary 2. *Let $u : A \to A'$ be a morphism and B be a finitely presented A-algebra. Then there exist a finitely presented A-algebra C and two A-morphisms $g : B \to C$, $w : C \to A'$ such that*

(1) *$v = wg$,*
(2) *$g(H_{B/A}) \subset H_{C/A}$,*
(3) *for every $b \in H_{B/A}$ the element $g(b)$ is standard for C over A.*

Proof. Let $C = S_A(I/I^2)$ be as in the above theorem, g the canonical inclusion and $h : C \to B$ the canonical retraction of g sending I/I^2 in 0. Then (2), (3) hold by the above theorem and $w = vh$ satisfies (1). $\square$

Set $h_B = \sqrt{v(H_{B/A})A'}$. We may suppose that $h_B \neq A'$. Let q be a minimal prime over ideal of h_B. After [5] we say that $A \to B \to A' \supset q$ is *resolvable* if there exists a finite type A-algebra C such that v factors through C, let us say $v = wg$, $g : B \to C$, $w : C \to A'$ and $h_B \subset h_C = \sqrt{w(H_{C/A})A'} \not\subset q$. The proof of the General Neron Desingularization follows from the next proposition.

Proposition 3. *Suppose that $A \supset \mathbf{Q}$, $p = u^{-1}q$ is a minimal over ideal of $u^{-1}(h_B)$, the map $A \to A'_q$ is flat and $A'q/pA'_q$ is a regular ring. Then $A \to B \to A' \supset q$ is resolvable.*

Assume that p is a minimal prime over ideal of $u^{-1}(h_B)$ and let $q_1, \dots, q_e$ be the minimal prime over ideals of h_b. Then p contains the product of $u^{-1}(q_i)$, $i \in [e]$ and so it contains one of them, let us say $u^{-1}(q)$ for $q = q_1$. The proof of the General Neron Desingularization follows applying the above proposition, which is possible because u is regular. Indeed, then $A \to B \to A' \supset q$ is resolvable and so $h_B \subset h_C \not\subset q$. It follows that $h_B \subsetneq h_C$. Applying several times this procedure by Noetherian induction we arrive to the case when $h_B = A'$.

Remark 4. Note that if q is minimal in A' in the assumption of Proposition 3 then $A_p \to A_p \otimes_A B \to A'_q \supset qA'_q$ is resolvable by the General Neron Desingularization on Artinian rings. In particular, $v_p : B_p = A_p \otimes_A B \to A'_q$ factors through a finite type A_p-algebra D such that $h_D A'_q = A'_q$. We may suppose that D is standard smooth.

Proposition 5. *Proposition 3 holds if q is minimal in A'.*

Proof. By Remark 4 $v_p : B_p = A_p \otimes_A B \to A'_q$ factors through a standard smooth A_p-algebra D, let us say $v_p = w_p g_p$ for some $g_p : B_p \to D$, $w_p : D \to A'_q$. We claim that we may choose D such that there exists a finite type B-algebra C with $A_p \otimes_A C \cong D$ and such that v factors through C, let us say $v = wg$, $w : C \to A'$, g being the canonical algebra structure morphism $B \to C$.

Indeed, let $D = B_p[Z_1, \ldots, Z_s]/(h)$, $h = (h_1, \ldots, h_e)$, w_p being given by $Z \to y/t$ for some $y \in A'^s$, $t \in A' \setminus q$. For large $l \in \mathbf{N}$ we get some homogeneous polynomials $h'_i(Y, T) = T^l h_i(Y/T)$, $Y = (Y_1, \ldots, Y_s)$ such that $h'_i(y, t) = 0$ in A'_q. This means that $rh'_i(y, t) = 0$ for some $t \in A' \setminus q$. Changing y with ry and t with rt we may suppose that $h'_i(y, t) = 0$ in A'. Let $C = B[Y, T]/(h')$ and $w : C \to A'$ be the extension of v by $Y \to y$, $T \to t$. Note that the B_p-morphism $D[T, T^{-1}] \to (B_p \otimes_B C)_T$ given by $Z \to Y/T$, $T \to T$ is an isomorphism and changing D by $D[T, T^{-1}]$ our claim is proved. It follows that $w(H_{C/A}) \not\subset q$, let us say $w(\alpha) \notin q$ for some $\alpha \in H_{C/A}$.

Remains to change C if necessary in order to have also $h_B \subset h_C$. Let $b_1, \ldots, b_\nu$ be a system of generators of $H_{B/A}$. If $\operatorname{Min} A' = \{q\}$ then $h_B = q = \sqrt{(0)}$ and there exists nothing to change because clearly $h_B \subset h_C$. Otherwise there exists $\gamma \in A' \setminus q$ such that $\gamma v(b_j)^k = 0$, $j \in [\nu]$ for some $k \in \mathbf{N}$. Set

$$C' = B[Y, T, (U_j)_{j \in [\nu]}, (V_{ij})_{j \in [\nu], i \in [e]}, \Gamma]/((h'_i + \sum_{j \in [\nu]} U_j V_{ij})_{i \in [e]}, (\Gamma U_j)_{j \in [\nu]}),$$

where U_j, V_{ij}, Γ are new variables. Note that v can be extended to $w' : C' \to A'$ given by $Y \to y$, $T \to t$, $U_j \to b_j^k$, $V_{ij} \to 0$, $\Gamma \to \gamma$ and $C'_\Gamma \cong C[(V_{ij})_{i,j}, \Gamma, \Gamma^{-1}]$ is a localization of a polynomial C-algebra. Thus $\Gamma\alpha \in H_{C'/A}$. On the other hand, $C'_{U_1} \cong B[Y, T, (U_j)_{j \in [\nu]}, U_1^{-1}, (V_{ij})_{1 < j \leq \nu, i \in [e]}]$ is a localization of a polynomial B-algebra. Thus $U_1 \in H_{C'/B}$ and so $v(b_1)^k \in w'(H_{C'/B})$. It follows that $v(b_1) \in h_{C'}$ and similarly $v(b_j) \in h_{C'}$ for all $j \in [\nu]$. Thus $h_B \subset h_{C'}$. Also note that $\gamma w(\alpha) \in h_{C'} \setminus q$. $\square$

Proposition 3 follows from the next lemma.

Lemma 6. *(Main Lemma* [4, Lemma 6.7]*) Let $a \in A$, $\bar{A} = A/a^8 A$, $\bar{B} = B/a^8 B$, $\bar{A}' = A'/a^8 A'$ and $\bar{q} = q/a^8 A'$. Suppose that*

(1) *$a \notin p$ for all $p \in \operatorname{Min} A$,*
(2) $\operatorname{Ann}_A(a^2) = \operatorname{Ann}_A(a)$, $\operatorname{Ann}_{A'}(u(a)^2) = \operatorname{Ann}_{A'}(u(a))$,
(3) *a is strictly standard for a certain presentation of B over A,*
(4) *$\bar{A} \to \bar{B} \to \bar{A}' \supset \bar{q}$ is resolvable.*

Then $A \to B \to A' \supset q$ is resolvable too.

Proof of Proposition 3. Apply induction on height of q. If $\operatorname{height} q = 0$ then apply Proposition 5. Assume that $\operatorname{height} q > 0$. If $\operatorname{height} p > 0$ then choose a in $u^{-1}(h_B)$ which lies in no minimal prime ideal of A. We have $\operatorname{height} p/(a) < \operatorname{height} p$, and so $\operatorname{height} q/u(a)A' < \operatorname{height} q$ because

$$\operatorname{height} q - \operatorname{height} p = \dim A'_q/pA'_q = \operatorname{height} q/u(a)A' - \operatorname{height} p/(a)$$

by flatness of $A_p \to A'_q$ (see [2, 15.1]). The chain $\operatorname{Ann}_A(a) \subset \ldots \subset \operatorname{Ann}_A(a^t) \subset$ stops by Noetherianity. Thus changing a by one of its powers we may suppose that $\operatorname{Ann}_A(a^2) = \operatorname{Ann}_A(a)$, $\operatorname{Ann}_{A'}(u(a)^2) = \operatorname{Ann}_{A'}(u(a))$. By Corollary 2 we may change B by another one finite type A-algebra and a by one of its power such that a is strictly standard for a certain presentation. Set $\bar{A} = A/a^8A$, $\bar{B} = B/a^8B$, $\bar{A}' = A'/a^8A'$ and $\bar{q} = q/a^8A'$. Then $\bar{A} \to \bar{B} \to \bar{A}' \supset \bar{q}$ is resolvable by induction hypothesis since $\operatorname{height} \bar{q} < \operatorname{height} q$ and it is enough to apply Main Lemma.

Now assume that $\operatorname{height} p = 0$. We show that we may reduce to the case when $\operatorname{height} p > 0$. Then A'_q/pA'_q is a regular local ring of dimension ≥ 1 and we may choose x in h_B inducing an element from the regular system of parameters of A'_q/pA'_q. Then the map $A[X]_(p, X) \to A'_q$, $X \to x$ is flat by the Local Flatness Criterion see [2]) since the map $k(p)[X]_(X) \to A'_q/pA'_q$ is flat by [2, Theorem 23.1]. By construction, $A'_q/(p, x)A'_q$ is still a regular local ring. Clearly, it is enough to show that $A[X] \to B[X] \to A' \supset q$ is resolvable. But now we have $\operatorname{height} p > 0$, that is the above case.

Main Lemma is a consequence of the following two lemmas.

Lemma 7. *(Lifting Lemma* [4, Lemma 7.1]*) Let $u : A \to A'$ be a morphism of Noetherian rings, $d \in A$, $\bar{A} := A/(d^2)$, $\bar{A}' := A'/d^2A'$, $\tilde{A} := A/(d)$, $\tilde{A}' := A'/dA'$, $\bar{C}$ a finite type $\bar{A}$-algebra and $\bar{\beta} : \bar{C} \to \bar{A}'$ a morphism of $\bar{A}$-algebras. Suppose that* $\operatorname{Ann}_A(d^2) = \operatorname{Ann}_A(d)$, $\operatorname{Ann}_{A'}(u(d)^2) = \operatorname{Ann}_{A'}(u(d))$. *Then there exist a finite type A-algebra D and a morphism $w : D \to A'$ of A-algebras such that*

(1) *$\tilde{A} \otimes_{\bar{A}} \bar{\beta}$ factors through $\tilde{A} \otimes_A w$,*
(2) *$\pi^{-1}(h_{\bar{C}}) \subset h_D$, π being the surjection $A' \to \bar{A}'$ and $h_{\bar{C}} = \sqrt{\bar{\beta}(H_{\bar{C}/\bar{A}})\bar{A}'}$.*

Lemma 8. *(Desingularization Lemma* [4, Lemma 7.2]*) Let $u : A \to A'$ be a morphism of Noetherian rings, B a finite type A-algebra and $v : B \to A'$ a morphism of A-algebras, $d \in A$, $\tilde{A} := A/(d^4)$, $\tilde{A}' := A'/d^4A'$, $\tilde{B} := B/d^4B$. Suppose that d is strictly standard for a certain presentation of B over A and* $\operatorname{Ann}_A(d^2) = \operatorname{Ann}_A(d)$, $\operatorname{Ann}_{A'}(u(d)^2) = \operatorname{Ann}_{A'}(u(d))$. *Let D be a finite type A-algebra and $w : D \to A'$ an A-morphism such that $\tilde{A} \otimes_A v$ factors through $\tilde{A} \otimes_A w$. Then there exist a finite type A-algebra E and an A-morphism $\gamma : E \to A'$ such that*

(1) *v, w factor through γ,*
(2) *$H_{D/A}E \subset H_{E/A}$, so $h_D \subset h_E$.*

Indeed, let $a \in A$, A', B be as in Main Lemma and set $d = a^4$. Since

$$\bar{A} := A/(d^2) \to \bar{B} := B/d^2B \to \bar{A}' := A'/d^2A' \supset \bar{q} := q/d^2A'$$

is resolvable there exist a finite type $\bar{A}$-algebra $\bar{C}$ and a morphism $\bar{\beta} : \bar{C} \to \bar{A}'$ of $\bar{A}$-algebras such that $h_{\bar{B}} \subset h_{\bar{C}} \not\subset \bar{q}$ and $\bar{v}$ factors through $\bar{\beta}$. We need a lifting D of $\bar{C}$ with $\pi^{-1}(h_{\bar{C}}) \subset h_D$. We cannot find this D with $\bar{A} \otimes_A D \cong \bar{C}$ or at least such that $\bar{\beta}$ factors through $\bar{A} \otimes_A D$. By Lifting Lemma there exist a finite type A-algebra D and an A-morphism $w : D \to A'$ such that $\tilde{A} \otimes_{\bar{A}} \bar{\beta}$ factors through $\tilde{A} \otimes_A w$ and and $\pi^{-1}(h_{\bar{C}}) \subset h_D$. Unfortunately, it is not clear that v factors through this D. So we try to find an A-algebra E such that v, w factor through E and $h_D \subset h_E$ By Desingularization Lemma applied for $d = a$ there exist a finite type A-algebra E and an A-morphism $\gamma : E \to A'$ such that v, w factor through γ and $h_D \subset h_E$. By base change $H_{B/A}\bar{B} \subset H_{\bar{B}/\bar{A}}$ and we get $h_B\bar{A}' \subset h_{\bar{B}} \subset h_{\bar{C}}$ and so $h_D \supset h_B$. We have also $h_D \not\subset q$ because $h_{\bar{C}} \not\subset \bar{q}$. Thus $h_B \subset h_E \not\subset q$, which is enough.

Remark 9. We believe that the above proof of the Main Lemma provides an algorithm to construct a finite type A-algebra E such that v factors through E and $h_B \subset h_E \not\subset q$. This is because D and E from the Lifting Lemma and the Desingularization Lemma could be really constructed. The proofs of these lemmas could be read in [4] and here we give only the constructions of D, E.

Construction of D

Choose $(\bar{P}_i)_{i\in[k]}$ strictly standard elements of $\bar{C}$ over $\bar{A}$ such that $H_{\bar{C}/\bar{A}} = \sqrt{(\bar{P}_i)_{i\in[k]})}$, let us say $\bar{P}_i \in \Delta_{\bar{f}^{(i)}}((\bar{f}^{(i)}) : \bar{I})$ for some presentation $\bar{C} = \bar{A}[X]/\bar{I}$, $X = (X_1, \dots, X_n)$, where $\bar{f}^{(i)}$ is a finite subsystem of $\bar{I}$. Let $f_1, \dots, f_s$ be polynomials in $A[X]$ such that $\bar{f}_j = f_j + d^2A[X]$, $1 \leq j \leq s$ generate $\bar{I}$ and $\{\bar{f}_1, \dots, \bar{f}_s\}$ contains all elements from $\bar{f}^{(i)}$. Set $I = (d^2, f_1, \dots, f_s)$ and let $\bar{\beta}$ be given by $X \to \bar{x} = x + d^2A'$ for some $x \in A'^n$. Then $f(x) = dz$ for some $z \in A'^s$. Set $g_j(X, Z) = f_j - dZ_j$, $1 \dots j \dots s$, $Z = (Z_1, \dots, Z_s)$ and $g = (g_j)$.

To every $\bar{P}_i$ we associate a system of polynomials $F^{(i)}$ in $A[X, Z]$ in the following way: Let P_i be a lifting of $\bar{P}_i$ to $A[X]$. Since $P_iI \subset (d^2, f_1, \dots, f_r)$ for a certain $r \leq s$ in a certain ordering of (f_j) depending on i, it follows that

$$P_i f_j = \sum_{i=1}^{r} H_{jt}^{(i)} f_t + d^2 G_j^{(i)},$$

for some polynomials $H_{jt}^{(i)}, G_j^{(i)}$ from $A[X]$, $r < j \leq s$. Set

$$F_j^{(i)} = P_i Z_j - \sum_{i=1}^{r} H_{jt}^{(i)} Z_t - dG_j^{(i)}, r < j \leq s, F^{(i)} = (F_j^{(i)})_{r<j\leq s}$$

and $D = A[X, Z]/(g, F^{(1)}, \dots, F^{(k)})$. Then D together with $w : D \to A'$, $(X, Z) \to (x, z)$ works.

Construction of E

Let $B = A[X]/I$, $X = (X_1, \dots, X_n)$, $S = B \otimes_A D = D[X]/I$ and $\alpha : S \to A'$ given by $b \otimes z \to v(b)w(z)$. Clearly, d is strictly standard also for S over D. Set $\tilde{A} = A/d^4A$. By the previous construction $\tilde{w} = \tilde{A} \otimes_A w$ and so $\tilde{v} = \tilde{A} \otimes_A v$ factors through $\tilde{D} = \tilde{A} \otimes_A D$, let us say $\tilde{v} = \tilde{w}\tilde{\tau}$ for some $\tilde{\tau} : \tilde{B} \to \tilde{D}$. Then the map

$\tilde{\rho} : \tilde{S} = \tilde{A} \otimes_A S \to \tilde{D}$ given by $\tilde{b} \otimes \tilde{z} \to \tilde{\tau}(\tilde{b})\tilde{z}$ is a retraction of $\tilde{D}$-algebras and $\tilde{w}\tilde{\rho} = \tilde{\alpha} = \tilde{A} \otimes_A \alpha$. Let $\tilde{\rho}$ be given by $X \to x + d^4 D^n$ for some $x \in D^n$.

Since d is strictly standard there exists a finite system of polynomials $f = (f_i)_{1 \leq i \leq r}$ in I such that $d \equiv P$ modulo I for a certain $P \in \Delta_f((f) : I)$ (in $D[X]$). We have $I(x) \equiv 0$ modulo d^4 and so $P(x) \equiv d$ modulo d^4 in D. Thus $P(x) = ds$ for a certain $s \in D$ with $s \equiv 1$ modulo d.

Let $P = \sum_\nu N_\nu M_\nu$, where M_ν are the $r \times r$-minors of $(\partial f/\partial X)$. If M_1 is given by the first columns we have $M_1 = \det H_1$, where H_1 is obtained adding down to $(\partial f/\partial X)$ as a border the block $(0|\mathrm{Id}_{n-r})$. Similarly define the n square matrices H_ν such that $\det H_\nu = M_\nu$. Clearly the first r rows of all H_ν coincide. Let G'_ν be the adjoint matrix of H_ν and $G_\nu = N_\nu G'_\nu$. We have

$$\sum_\nu G_\nu H_\nu = \sum_\nu N_\nu M_\nu \mathrm{Id}_n = P\mathrm{Id}_n$$

and so

$$ds\mathrm{Id}_n = P(x)\mathrm{Id}_n = \sum_\nu G_\nu(x) H_\nu(x).$$

Let α be given by $X \to y \in A'^n$. Set $x' = w(x)$. We have $y - x' \in d^4 A'^n$, let us say $y - x' = d^3 \varepsilon$ for $\varepsilon \in dA'^n$. Then $t^{(\nu)} := H_\nu(x)\varepsilon$ satisfies

$$\sum_\nu G_\nu(x) t^{(\nu)} = P(x)\varepsilon = ds\varepsilon$$

and so

$$s(y - x') = d^2 \sum_\nu G_\nu(x) t^{(\nu)}.$$

As the first r rows of H_ν coincide we see that $t^{(\nu)} = (u_1, \ldots, u_r, t^{(\nu)}_{r+1}, \ldots, t^{(\nu)}_n)$, where u_i are independent of ν. Let

$$h = s(X - x) - d^3 W - d^2 \sum_\nu G_\nu(x) T^{(\nu)},$$

where $W = (W_1, \ldots, W_n)$ and $T^{(\nu)} = (U_1, \ldots, U_r, T^{(\nu)}_{r+1}, \ldots, T^{(\nu)}_n)$ are new variables and $T = (T^{(\nu)})$. The kernel of the map $\varphi : D[X, T, W] \to A'$ given by $X \to y$, $T^{(\nu)} \to t^{(\nu)}$, $W \to 0$ contains h. Since

$$s(X - x) \equiv d^3 W + d^2 \sum_\nu G_\nu(x) T^{(\nu)} \text{ modulo } h$$

and

$$f(X) - f(x) \equiv \sum_j \partial f/\partial X_j(x)(X_j - x_j)$$

modulo higher order terms in $X_j - x_j$ by Taylor's formula we see that for $m = \max_i \deg f_i$ we have

$$s^m f(X) - s^m f(x) \equiv \sum_j s^{m-1} \partial f/\partial X_j(x)[d^3 W_j + d^2 \sum_\nu G_\nu(x) T^{(\nu)}_j] + d^4 Q'$$

modulo h where $Q' \in D[W,T]^r$. Set $Q = dQ' + \sum_j s^{m-1}\partial f/\partial X_j(x)W_j$. We have $f(x) = d^3c$ for some $c \in dD^r$. It is not hard to see that $g = s^m c + s^m U + Q$ is in the kernel of φ. Set $E = D[X,W,V]/(I,g,h)$ and let γ be the map induced by φ.

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EXTRA LUMINY 2. APPROXIMATION IN NESTED SUBRINGS AND STRONG ARTIN APPROXIMATION

DORIN POPESCU

Lemma 3 below was done by Guillaume in the case when some solutions are in $\hat{A}[x]$ and not in $\hat{A}[x]^h$. It is easier to understand and I think you should follow his proof.

Let (A, m, k) be a local ring. An essentially etale local A-algebra B is an *etale neighborhood* of A if $B \otimes_A k \cong k$.

Theorem 1. *([7]) A local A-algebra B is an etale neighborhood of A if and only if $B \cong (A[T]/(g))_{(m,T)}$, g being a monic polynomial in T over A such that $g(0) \in m$, $\partial g/\partial T(0) \notin m$.*

Theorem 2. *([7]) The filtered colimit of all etale neighborhood of A is a flat local A-algebra A^h with the property that for every Henselian local A-algebra C there exists an unique local A-morphism $A^h \to B$. If A is Noetherian then A^h is too.*

The ring A^h defined above is the *Henselization* of A.

Lemma 3. *Let $\hat{A}$ be the completion of A, $A[x]^h$, $x = (x_1, \ldots, x_n)$, $\hat{A}[x]^h$ be the Henselizations of $A[x]_{(m,x)}$ respectively $\hat{A}[x]_{(m,x)}$, $f = (f_1, \ldots, f_r)$ a system of polynomials in $Y = (Y_1, \ldots, Y_N)$ over $A[x]^h$ and $1 \leq t < N$ and c be some positive integers. Suppose that A has the property of approximation and f has a solution $\hat{y} = (\hat{y}_1, \ldots, \hat{y}_N)$ in $\hat{A}[x]^h$ such that $\hat{y}_i \in \hat{A}$ for all $i \leq t$. Then there exists a solution $y = (y_1, \ldots, y_N)$ of f in $A[x]^h$ such that $y_i \in A$ for all $i \leq t$ and $y \equiv \hat{y} \mod m^c\hat{A}[x]^h$.*

Proof. Take an etale neighborhood B of $\hat{A}[x]_{(m,x)}$ such that $\hat{y}_i \in B$ for all $t < i \leq N$. Then $B \cong (\hat{A}[x, T]/(\hat{g}))_{(m,x,T)}$ for some monic polynomial $\hat{g}$ in T over $\hat{A}[x]$ with $\hat{g}(0) \in (m, x)$ and $\partial\hat{g}/\partial T(0) \notin (m, x)$, let us say $\hat{g} = T^e + \sum_{j=0}^{e-1}(\sum_{k\in\mathbf{N}^n,|k|<u} \hat{z}_{jk}x^k)T^j$, for some u high enough and $\hat{z}_{jk} \in \hat{A}$. Note that $\hat{z}_{00} \in m\hat{A}$ and $\hat{z}_{10} \notin m\hat{A}$. We suppose that $\hat{y}_i \equiv \sum_{j=0}^{e-1}(\sum_{k\in\mathbf{N}^n,|k|<u} \hat{y}_{ijk}x^k)T^j \mod \hat{g}$ for some $\hat{y}_{ijk} \in \hat{A}$, $t < i \leq N$. Actually, we should take $\hat{y}_i$ as a fraction but for an easier expression we will skip the denominator. Substitute $Y_i^+ = \sum_{j=0}^{e-1}(\sum_{k\in\mathbf{N}^n,|k|<u} Y_{ijk}x^k)T^j$ in f and divide by the monic polynomial $G = T^e + \sum_{j=0}^{e-1}(\sum_{k\in\mathbf{N}^n,|k|<u} Z_{jk}x^k)T^j$ in $\hat{A}[x, T, Y_1, \ldots, Y_t, (Y_{ij}), (Z_j)]$, where $(Y_{ijk}), (Z_{jk})$ are new variables.

We get

$$f_p(Y_1, \ldots, Y_t, Y^+) \equiv \sum_{j=0}^{e-1}(\sum_{k\in\mathbf{N}^n,|k|<u} F_{pjk}(Y_1, \ldots, Y_t, (Y_{ijk}), (Z_{jk}))x^kT^j \mod G,$$

$1 \leq p \leq r$. Then $\hat{y}$ is a solution of f in B if and only if $\hat{y}_1, \ldots, \hat{y}_t, (\hat{y}_{ijk}), (\hat{z}_{jk}))$ is a solution of (F_{pjk}) in $\hat{A}$. As A has the property of approximation we may choose a solution $(y_1,\ldots,y_t,(y_{ijk}),(z_{jk}))$ of (F_{pjk}) in A which coincide modulo $m^c\hat{A}$ with the former one. Then $y_i = \sum_{j=0}^{e-1}(\sum_{k\in\mathbf{N}^n,|k|<u}(y_{ijk}))x^kT^j$, $t<i$ together with y_i, $i\leq t$ form a solution of f in the etale neighborhood $B' = (A[x,T]/(g))_{(m,x,T)}$, $g = T^e + \sum_{i=0}^{e-1}(\sum_{k\in\mathbf{N}^n,|k|<u} z_{jk}x^k)T^j$ of $A[x]_{(m,x)}$, which is contained in $A[x]^h$. Clearly, y is the wanted solution. □

The following theorem generalizes the above lemma in the same idea. The proof does not go by recurrence using the above lemma (see [6, Proposition 3.5] for a proof).

Proposition 4. *In the setting of the above lemma, let $0\leq s_1\leq\ldots\leq s_N\leq n$ and c be some non-negative integers. Suppose that A has the property of approximation and f has a solution $\hat{y}=(\hat{y}_1,\ldots,\hat{y}_N)$ in $\hat{A}[x]^h$ such that $\hat{y}_i\in\hat{A}[x_1,\ldots,x_{s_i}]^h$ for all $1\leq i\leq N$. Then there exist a solution $y=(y_1,\ldots,y_N)$ of f in $A[x]^h$ such that $y_i\in A[x_1,\ldots,x_{s_i}]^h$ for all $1\leq i\leq N$ and $y\equiv\hat{y} \mod m^c\hat{A}[x]^h$.*

The idea of the following theorem as well as of the above proposition we got in 1977 from H. Kurke and G. Pfister.

Theorem 5. *(Popescu* [5], [6, Theorem 3.6]*) Let (A,m) be an excellent Henselian local ring, $\hat{A}$ be the completion of A, $A[x]^h$, $x=(x_1,\ldots,x_n)$, be the Henselization of $A[x]_{(m,x)}$, $f=(f_1,\ldots,f_r)$ be a system of polynomials in $Y=(Y_1,\ldots,Y_N)$ over $A[x]^h$ and $0\leq s_1\leq\ldots\leq s_N\leq n$, c be some non-negative integers. Suppose that f has a solution $\hat{y}=(\hat{y}_1,\ldots,\hat{y}_N)$ in $\hat{A}[[x]]$ such that $\hat{y}_i\in\hat{A}[[x_1,\ldots,x_{s_i}]]$ for all $1\leq i\leq N$. Then there exists a solution $y=(y_1,\ldots,y_N)$ of f in $A[x]^h$ such that $y_i\in A[x_1,\ldots,x_{s_i}]^h$ for all $1\leq i\leq N$ and $y\equiv\hat{y} \mod (m,x)^c\hat{A}[[x]]$.*

Proof. We will study the particular case when $s_i=0$ for $1\leq i\leq t$ and $s_i=n$ for $t<i\leq n$ for some $1\leq t<N$, that is the case of Lemma 3. Then $\hat{A}[x]^h$ is execellent Henselian and so it has the property of Artin approximation. Thus the polynomial equations $f(\hat{y}_1,\ldots,\hat{y}_t,Y_{t+1},\ldots,Y_N)$ from $\hat{A}[Y_{t+1},\ldots,Y_N]$ have a solution $(\tilde{y}_{t+1},\ldots,\tilde{y}_N)$ in $\hat{A}[x]^h$ such that $\tilde{y}_i\equiv\hat{y}_i \mod (m,x)^c\hat{A}[[x]]$. By Lemma 3 we find for the solution $(\hat{y}_1,\ldots,\hat{y}_t,\tilde{y}_{t+1},\ldots,\tilde{y}_N)$ of f in $\hat{A}[x]^h$ a solution y of f in $A[x]^h$ such that $y_i\in A$ for $1\leq i\leq t$ which coincide with the previous one modulo $m^cA[x]^h$. The general case goes by recurrence using the above proposition instead Lemma 3. □

Corollary 6. *Let K be a field, $A=K<x>$, $x=(x_1,\ldots,x_n)$, $f=(f_1,\ldots,f_r)$ be a system of polynomials in $Y=(Y_1,\ldots,Y_N)$ over A and $0\leq s_1\leq\ldots\leq s_N\leq n$, c be some non-negative integers. Suppose that f has a solution $\hat{y}=(\hat{y}_1,\ldots,\hat{y}_N)$ in $K[[x]]$ such that $\hat{y}_i\in K[[x_1,\ldots,x_{s_i}]]$ for all $1\leq i\leq N$. Then there exists a solution $y=(y_1,\ldots,y_N)$ of f in $K<x>$ such that $y_i\in K<x_1,\ldots,x_{s_i}>$ for all $1\leq i\leq N$ and $y\equiv\hat{y} \mod (m,x)^cK[[x]]$.*

The above corollary answers positively a question of M. Artin (see [1]). An attempt to prove it was done in [3].

Corollary 7. *The Weierstrass Preparation Theorem holds for the algebraic power series over a field.*

Proof. Let $f \in K < x >$, $x = (x_1, \dots, x_n)$ be an algebraic power series such that $f(0, \dots, 0, x_n) \neq 0$. By Weierstrass Preparation Theorem f is associated in divisibility with a monic polynomial $\hat{g} = x_n^u + \sum_{i=0}^{u-1} \hat{z}_i x_n^i \in K[[x_1, \dots, x_{n-1}]][x_n]$ for some $\hat{z}_i \in (x_1, \dots, x_{n-1})K[[x_1, \dots, x_{n-1}]]$. Thus the system $F_1 = f - Y(x_n^u + \sum_{i=0}^{u-1} Z_i x_n^i)$, $F_2 = YU - 1$ has a solution $\hat{y}$, $\hat{u}$, $\hat{z}_i$ in $K[[x]]$ such that $\hat{z}_i \in K[[x_1, \dots, x_{n-1}]]$. By Corollary 6 there exists a solution y, u, z_i in $K < x >$ such that
$z_i \in K < x_1, \dots, x_{n-1} >$ which is congruent modulo (x) with the previous one. Thus y is invertible and $f = yg$, where $g = x_n^u + \sum_{i=0}^{u-1} z_i x_n^i \in K < x_1, \dots, x_{n-1} > [x_n]$. By unicity of the (formal) Weierstrass Preparation Theorem we get in fact $u = \hat{u}$, $y = \hat{y}$, $z_i = \hat{z}_i$ and so $g = \hat{g}$. □

The idea to apply ultrapower methods to the strong Artin approximation comes from [2] (see also [4]). We start with some preparations. Let D be a *filter* on $\mathbf{N}$, that is a family of subsets of $\mathbf{N}$ satisfying

1) $\emptyset \notin D$,
2) if $s, t \in D$ then $s \cap t \in D$,
3) if $s \in D$, $s \subset t \subset \mathbf{N}$ then $t \in D$.

An *ultrafilter* is a maximal filter in the set of filters on $\mathbf{N}$ with respect to the inclusion. A filter D on $\mathbf{N}$ is an ultrafilter if and only if $\mathbf{N} \setminus \mathbf{s} \in \mathbf{D}$ for all $s \subset \mathbf{N}$ which is not in D. An ultrafilter is *nonprincipal* if it contains the filter of all cofinite subsets of $\mathbf{N}$.

The *ultrapower* A^* of a ring A with respect to a nonpricipal ultrafilter D is the quotient of $A^{\mathbf{N}}$ by the ideal I_D of all $(x_n)_{n \in \mathbf{N}})$ such that the set $\{n \in \mathbf{N} : \mathbf{x_n} = \mathbf{0}\} \in \mathbf{D}$. Denote by $[(x_n)]$ the class modulo I_D of all $(x_n) \in A^{\mathbf{N}}$. Assigning to $a \in A$ the constant sequence $[(a, a, \dots)]$ we get a ring morphism $\varphi_A : A \to A^*$.

Lemma 8. *Suppose that (A, m) is a Noetherian local ring. The following statements hold:*

(1) *If A is a field then A^* is a field and the field extension A^*/A is separable,*
(2) *A^* is a local ring with $\varphi(m)A^*$ its maximal ideal,*
(3) *If A is Henselian then A^* is Henselian too,*
(4) *A^* is not Noetherian if A is not Artinian,*
(5) *The separation $A_1 = A^*/m_\infty$, $m_\infty = \cap_{j \in \mathbf{N}} \varphi(m^j)A^*$ of A^* in the m-adic topology is a complete Noetherian local ring,*
(6) *φ and the composite map $u : A \xrightarrow{\varphi} A^* \to A_1$ are flat,*
(7) *If A is excellent then u is regular.*

The proof of this lemma is given on pages 328-331 of [6]. Next proposition shows the connection of ultraproducts with the strong Artin approximation.

Proposition 9. *The following statements are equivalent:*

(1) *A has strong Artin Approximation,*

(2) *For every finite system of polynomial equations f over A, for every positive integer c and for every solution solution $\tilde{y}$ of f in A^* modulo m_∞ there exists a solution y of f in A^* such that $y \equiv \tilde{y}$ modulo $m^c A^*$.*

Proof. $(i) \Rightarrow (ii)$. Let f, $\tilde{y} = [(y_n)]$ be like in (ii) and $\nu : \mathbf{N} \to \mathbf{N}$ the Artin function associated to f. In particular we have $f(\tilde{y}) \equiv 0 \bmod m^{\nu(c)}A^*$. Thus the set $s = \{n \in \mathbf{N} : \mathbf{f}(\mathbf{\tilde{y}_n}) \equiv \mathbf{0} \bmod \mathbf{m}^{\nu(\mathbf{c})} \in \mathbf{D}\}$. Then for every $n \in s$ there exists a solution y_n of f in A such that $y_n \equiv \tilde{y}_n \bmod m^c$. Set $y_n = 0$ for $n \notin s$. Then $y = [(y_n)]$ is a solution of f in A^* such that $y \equiv \tilde{y} \bmod m^c A^*$.

$(ii) \Rightarrow (i)$. Assume that there exists a finite system of polynomials f in some variables Y over A which has no Artin function; that is there exists a positive integer c such that

$(*)$ For every $n \in \mathbf{N}$ there exists $\tilde{y}_n$ in A such that $f(\tilde{y}_n) \equiv 0 \bmod m^n$ but there exists no solution y'_n of f in A such that $y'_n \equiv \tilde{y}_n \bmod m^c$.

Then $\tilde{y} = [(\tilde{y}_n)]$ is a solution of f in A^* mod $m^r A^*$ for all $r \in \mathbf{N}$. Thus $f(\tilde{y}) \equiv 0 \bmod m_\infty$. By (ii) there exists a solution y of f in A^* such that $y \equiv \tilde{y}$ modulo $m^c A^*$. Then the set $s = \{n \in \mathbf{N} : \mathbf{f}(\mathbf{y_n}) = \mathbf{0}, \mathbf{y_n} \equiv \mathbf{\tilde{y}_n} \bmod\} \in \mathbf{D}$ is nonempty. Note that y_n for some $n \in s$ contradicts $(*)$. □

Theorem 10. *(Popescu* [5], [6]*) An excellent Henselian local ring has the property of strong approximation.*

Proof. Let (A, m) be an excellent Henselian local ring, D be a nonprincipal ultrafilter on $\mathbf{N}$, A^* the ultrafilter of A with respect to D, and $u : A \to A_1$ the regular morphism defined in Lemma 8 (7). By Proposition 9 it is enough to show that given a system of polynomial equations h in $Z = (Z_1, \ldots, Z_s)$ over A, a positive integer c and $\tilde{z}$ a solution of h in A^* modulo m^*_∞ there exists a solution z of h in A^* such that $z \equiv \tilde{z} \mod m^c A^*$. By General Neron Desingularization applied to ψ_A the A-morphism $v : B = A[Z]/(h) \to A_1$, $Z \to \tilde{z} \mod m^*_\infty$ factors through a smooth A-algebra of type $C = (A[Y]/(f))_g$ $Y = (Y_1, \ldots, Y_N)$, where $f = (f_1, \ldots, f_r)$, $r \leq N$, are polynomials in Y over A and g belongs to the ideal Δ_f generated by all $r \times r$-minors of the Jacobian matrix $(\partial f_i / \partial Y_j)$, let us say $v = wq$, $w : C \to A_1$, $q : B \to C$. Then $\hat{y} = w(\hat{Y})$ is a solution of f in A_1 such that $g(\hat{y}) = w(\hat{g}) \notin mA_1$. Let $\tilde{y}$ be a lifting of $\hat{y}$ to A^*. In particular, $f(\tilde{y}) \equiv 0 \mod m^c A^* = g^2(\tilde{y}) m^c A^*$. But A^* is a Henselian local ring by Lemma 8 (3) and so by the Implicit Function Theorem we get a solution y of f in A^* such that $y \equiv \tilde{y} \mod m^c A^* = g(\tilde{y}) m^c A^*$. Then we get an A-morphism $u : C \to A^*$ by $Y \to y$. Clearly, $z = uq(\hat{Z})$ is a solution of h in A^* such that $z \equiv \tilde{z} \mod m^c A^*$. □

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ARTIN APPROXIMATION, VERSAL DEFORMATIONS AND MAXIMAL COHEN-MACAULAY MODULES

DORIN POPESCU

In November 1980 I visited MIT, where Prof. M. Artin showed me the following theorem.

Theorem 1. *(Ploski* [5]*) Let* $\mathbf{C}\{x\}$, $x = (x_1, \ldots, x_n)$, $f = (f_1, \ldots, f_s)$ *be some convergent power series from* $\mathbf{C}\{x, Y\}$, $Y = (Y_1, \ldots, Y_N)$ *and* $\hat{y} \in \mathbf{C}[[x]]^N$ *with* $\hat{y}(0) = 0$ *be a solution of* $f = 0$. *Then the map* $v : B = \mathbf{C}\{x, Y\}/(f) \to \mathbf{C}[[x]]$ *given by* $Y \to y$ *factors through an* A*-algebra of type* $B' = \mathbf{C}\{x, Z\}$ *for some variables* $Z = (Z_1, \ldots, Z_s)$, *that is* v *is a composite map* $B \to B' \to \mathbf{C}[[x]]$.

This result showed me that it is possible to get a kind of Neron Desingularization in dimension > 1, and gave me power to prove later the following theorem.

Theorem 2. *(General Neron Desingularization, Popescu* [6], [7]*, Swan* [12]*, Spivakovski* [11]*) Let* $u : A \to A'$ *be a regular morphism of Noetherian rings and* B *a finite type* A*-algebra. Then any* A*-morphism* $v : B \to A'$ *factors through a smooth* A*-algebra* C, *that is* v *is a composite* A*-morphism* $B \to C \to A'$.

H. Hauser asks me if this theorem does not imply somehow Ploski's result at least in the case when f are polynomials in Y. My positive answer is the following theorem.

Theorem 3. *Let* (A, m) *be an excellent Henselian local ring,* $\hat{A}$ *its completion,* B *a finite type* A*-algebra and* $v : B \to \hat{A}$ *an* A*-morphism. Then* v *factors through an* A*-algebra of type* $A < Z >= A[Z]^h$ *for some variables* $Z = (Z_1, \ldots, Z_s)$, *that is* $A < Z >$ *is the Henselization of* $A[Z]_{(m,Z)}$.

Proof. By Theorem 2 we see that v factors through a smooth A-algebra B', let us say v is the composite map $B \to B' \xrightarrow{v'} \hat{A}$. Using the local structure of smooth algebras given by Grothendieck we may assume that a $B'_{v'^{-1}(m\hat{A})}$ is a localization of a smooth A-algebra of type $(A[Z, T]/(g))_{g'h}$, where $Z = (Z_1, \ldots, Z_s)$, $g' = \partial g/\partial T$. Choose h such that v' factors through $C = (A[Z, T]/(g))_{g'h}$ let us say v' is the composite map $B' \to C \xrightarrow{w} \hat{A}$.

Suppose that w is given by $(Z, T) \to (\hat{z}, \hat{t}) \in \hat{A}$. We claim that we may reduce to the case when $\hat{z}(0) = 0$, $\hat{t}(0) = 0$. Indeed, choose $z_0 \in A^N$, $t_0 \in A$, such that $(z_0, t_0) \equiv (\hat{z}, \hat{t})$ modulo $m\hat{A}$ and set $\hat{z}' = \hat{z} - z_0 \in m\hat{A}$, $\hat{t}' = \hat{t} - t_0 \in m\hat{A}$. Changing (Z, T) by $(z_0 + Z', t_o + T')$, $Z' = (Z'_1, \ldots, Z'_N)$ in C and correspondingly $(\hat{z}, \hat{t})$ by $(z_0 + \hat{z}', t_0 + \hat{t})$ in $\hat{A}$ we get our claim fulfilled.

Clearly w extends to a map $w' : C' = (A < Z > [T]/(g))_{g'h} \to \hat{A}$ and we have $C' \cong A < Z >$ since C' is an etale neighborhood of $A < Z >$. $\square$

The following theorem was conjectured by M. Artin in [2] and it is a consequence of Theorem 2.

Theorem 4. *(Popescu* [6], [9]*) An excellent Henselian local ring has the property of Artin approximation.*

This theorem follows easily from Theorem 2 using the Implicit Function Theorem. But it is much easier to apply Theorem 3. Indeed, let (A, m) be an excellent Henselian local ring, $f = (f_1, \ldots, f_r)$ some polynomials from $A[Y]$, $Y = (Y_1, \ldots, Y_N)$ and $\hat{y} \in \hat{A}^N$ a solution of $f = 0$. We will show that f has a solution in A. Set $B = A[Y]/(f)$ and $v : B \to \hat{A}$ the map given by $Y \to \hat{y}$. By Theorem 3 v factors through $A < Z >$ for some $Z = (Z_1, \ldots, Z_s)$, that is v is the composite map $B \xrightarrow{g} A < Z > \to \hat{A}$. Set $y = g(Y + (f))$. Then $y(0)$ is a solution of f in A.

Theorem 5. *(Popescu* [8]*) Let (A, m) be a Noetherian local ring with the completion map $A \to \hat{A}$ regular. Then for every finite type A-algebra B there exists a function $\lambda : \mathbf{N} \to \mathbf{N}$ such that for every positive integer c and every morphism $v : B \to A/m^{\lambda(c)}$ there exists a smooth A-algebra C and two A-algebra morphisms $t : B \to C$, $w : C \to A/m^c$ such that wt is the composite map $B \xrightarrow{v} A/m^{\lambda(c)} \to A/m^c$.*

Remark 6. Can be λ computed? The proof from [8] it is not constructive. Perhaps λ can be related with the Artin function associated to f, which is defined by $B = A[Y]/(f)$. More precisely, λ could be related with the Artin function of f with respect to $\hat{A}$, or $A < Z >$, $Z = (Z_1, \ldots, Z_s)$. I do not know to choose this s.

Let (A, m) be an excellent Henselian local ring, $\hat{A}$ its completion and MCM(A) (resp. MCM$(\hat{A})$) the set of isomorphism classes of maximal Cohen Macaulay modules over A (resp. $\hat{A}$). Assume that A is an isolated singularity. Then a maximal Cohen-Macaulay module is free on the punctured spectrum. Since $\hat{A}$ is also an isolated singularity we see that the map φ :MCM(A) $\to$MCM$(\hat{A})$ given by $M \to \hat{A} \otimes_A M$ is surjective by a theorem of Elkik [3, Theorem 3].

Theorem 7. *(Popescu-Roczen,* [10]*) φ is bijective.*

Corollary 8. *In the hypothesis of the above theorem if $M \in MCM(A)$ is indecomposable then $\hat{A} \otimes_A M$ is indecomposable too.*

Proof. Assume that $\hat{A} \otimes_A M = \hat{N}_1 \oplus \hat{N}_2$. Then $\hat{N}_i \in$ MCM$(\hat{A})$ and by surjectivity of φ we get $\hat{N}_i = \hat{A} \otimes_A N_i$ for some $N_i \in$ MCM(A). Then $\hat{A} \otimes_A M \cong (\hat{A} \otimes_A N_1) \oplus (\hat{A} \otimes_A N_2)$ and the injectivity of φ gives $M \cong N_1 \oplus N_2$. $\square$

Remark 9. If A is not Henselian then the above corollary is false. For example let $A = \mathbf{C}[\mathbf{X}, \mathbf{Y}]_{(\mathbf{X},\mathbf{Y})}/(\mathbf{Y^2} - \mathbf{X^2} - \mathbf{X^3})$. Then $M = (X, Y)A$ is indecomposable in MCM(A) but $\hat{A} \otimes_A M$ is decomposable. Indeed, for $\hat{u} = \sqrt{1 + X} \in \hat{A}$ we have $\hat{A} \otimes_A M = (Y - \hat{u}X)\hat{A} \oplus (Y - \hat{u}X)\hat{A}$.

Remark 10. Let $\Gamma(A)$, $\Gamma(\hat{A})$ be the so called the AR-quivers of A, $\hat{A}$. Then φ induces also an inclusion $\Gamma(A) \subset \Gamma(\hat{A})$ (see [10]).

Remark 11. It is known that $\mathrm{MCM}(\hat{A})$ is finite if and only if $\hat{A}$ is a simple singularity. What about a complex unimodal singularity R? Certainly in this case $\mathrm{MCM}(R)$ is infinite but may be it has a special property, which characterizes the unimodal singularities. For this purpose it would be necessary to describe somehow $\mathrm{MCM}(R)$ at least in some special cases. Small attempts are done by Andreas Steenpass in his PhD thesis.

Next, we see how to get algebraic versal deformations using a special form of Artin approximation. Let $D = K < Z >$, $A = K < T > /J$, $Z = (Z_1, \ldots, Z_s)$, $T = (T_1, \ldots, T_n)$ and $N = D/(f_1, \ldots, f_d)$. A deformation of N over A is a $P = K < T, Z > /(J) \cong (A \otimes_K D)^h$-module L such that

1) $L \otimes_A K \cong N$,

2) L is flat over A,

where above B^h denotes the Henselization of a local ring B. The condition 1) says that L has the form $P/(F_1, \ldots, F_d)$ with $F_i \in K < T, Z >$, $F_i \cong f_i$ modulo (T) and 2) says that

2′) $\mathrm{Tor}_1^A(L, K) = 0$

by the Local Flatness Criterion since L is (T)-adically ideal separated because P is local Noetherian. Let $P^e \xrightarrow{\nu} P^d \to P \to L \to 0$ be part of a free resolution of L over P, where the map $P^d \to P$ is given by $(F_1, \ldots, F_d)$. Then 2′) says that tensorizing with $K \otimes_A -$ the above sequence we get an exact sequence $D^e \to D^d \to D \to N \to 0$ because P is flat over A. Therefore, 2′) is equivalent with

2″) For all $g \in D^d$ with $\sum_{i=1}^d g_i f_i = 0$ there exists $G \in K < T, Z >^d$ with $G \equiv g$ modulo (T) such that G modulo $J \in \mathrm{Im}\, \nu$, that is $\sum_{i=1}^d G_i F_i \in (J)$.

We would like to construct a versal deformation L (see [4, pages 157-159]), that is for any $A' = K < U > /J'$, $U = (U_1, \ldots, U_{n'})$, $P' = (A' \otimes_K D)^h$ and $L' = P'/(F')$ a deformation of N to A' there exists a morphism $\alpha : A \to A'$ such that $P' \otimes_P L \cong L'$, where the structural map of P' over P is given by α. If we replace above the algebraic power series with formal power series then this problem is solved by Schlessinger in the infinitesimal case followed by some theorems of Elkik and M. Artin. Set $\hat{A} = K[[T]]/(J)$, $\hat{P} = (\hat{A} \otimes_K D)^h$. We will assume that we have already L such that $\hat{L} = \hat{P} \otimes_P L$ is versal in the frame of complete local rings. How to get the versal property for L in the frame of algebraic power series?

Let A', P', L' be as above. Since $\hat{L}$ is versal in the frame of complete local rings there exists $\hat{\alpha} : \hat{A} \to \hat{A}'$ such that $\hat{P}' \otimes_{\hat{P}} \hat{L} \cong \hat{L}' = \hat{P}' \otimes_{P'} L'$, where the structure of $\hat{P}'$ as a $\hat{P}$-algebra is given by $\hat{\alpha}$. Assume that $\hat{\alpha}$ is given by $T \to \hat{t} \in (U)K[[U]]^n$. Then we have

i) $J(\hat{t}) \equiv 0$ modulo (J').

On the other hand we may suppose that $\hat{\alpha}$ induces an isomorphism $\hat{P}' \otimes_{\hat{P}} \hat{L} \to \hat{L}'$ which is given by $(T, Z) \to (\hat{t}, \hat{z})$ for some $\hat{z} \in (U, Z)K[[U, Z]]^s$ with $\hat{z} \equiv Z$ modulo $(U, Z)^2$ and the ideals $(F(\hat{t}, \hat{z}))$, (F') of $K[[U, Z]]$ coincide. Thus there exists an invertible $d \times d$-matrix $\hat{C} = (\hat{C}_{ij})$ over $K[[U, Z]]$ with

ii) $F'_i = \sum_{j=1}^d \hat{C}_{ij} F_j(\hat{t}, \hat{z}))$.

In this moment we would like to apply a special form of Artin approximation namely the so called Artin approximation with nested subring condition.

Theorem 12. *(Popescu [6], [9]) Let K be a field, $A = K < x >$, $x = (x_1, \ldots, x_n)$, $f = (f_1, \ldots, f_r) \in K < x, Y >^r$, $Y = (Y_1, \ldots, Y_N)$ and $0 \leq s \leq n$, $1 \leq t \leq N$, c be some non-negative integers. Suppose that f has a solution $\hat{y} = (\hat{y}_1, \ldots, \hat{y}_N)$ in $K[[x]]$ such that $\hat{y}_i \in K[[x_1, \ldots, x_s]]$ for all $1 \leq i \leq t$. Then there exists a solution $y = (y_1, \ldots, y_N)$ of f in $K < x >$ such that $y_i \in K < x_1, \ldots, x_s >$ for all $1 \leq i \leq t$ and $y \equiv \hat{y} \mod (x)^c K[[x]]$.*

By the above theorem we may find $t \in (U)K < U >^n$ and $z \in (U, Z)K < U, Z >^s$, $C_{ij} \in K < U, Z >$ satisfying i), ii) and such that $t \equiv \hat{t}$, $z \equiv \hat{z}$, $C_{ij} \equiv \hat{C}_{ij}$ modulo $(U, Z)^2$. Note that $\det(C_{ij}) \equiv \det \hat{C}$ modulo $(U, Z)^2$ and so (C_{ij}) is invertible. It follows that $\alpha : A \to A'$ given by $T \to t$ is the wanted one, that is $P' \otimes_P L \cong L'$, where the structure of P' as a P-algebra is given by α.

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