Three lectures on Constructive Algebra

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1. Structure of finitely generated abelian groups

The first lecture introduces some main features of constructive algebra w.r.t. classical algebra through the example of the structure theorem describing finitely generated abelian groups, and its generalization to finitely generated modules over a PID.

The true constructive content of these theorems is the Smith’ algorithm for reducing matrices to a convenient normal form.

The theory of matrices over a ring appears also in the theory of solutions of linear systems with coefficients and unknowns in the given ring.

The notion of coherent ring (a ring in which finitely generated ideals are finitely presented) appears as the crucial notion for managing linear systems over a commutative ring.

In classical mathematics this notion is almost everywhere hidden behind the notion of Noetherianity. We explore the interplay between coherence and Noetherianity from the constructive point of view.

2. Constructive aspects of Krull dimension

Krull dimension is a central concept in commutative algebra. It appears in the hypotheses of many “great theorems”.

Based on chains of prime ideals, the notion of Krull dimension of a ring should appear strongly as a nonconstructive one.

Nevertheless it is possible to give in classical mathematics an equivalent definition that uses only quantifications over elements of the ring (without speaking of prime ideals).

We explain why this new “elementary” definition of the Krull dimension allows us to manage in an algorithmic way many (and hopefully all) “great theorems” that use Krull dimension in classical mathematics.

This is a striking example showing that many classical theorems that appear a priori out of the scope of constructive mathematics can in fact be deciphered as hiding in their “abstract” proofs concrete algorithms for constructing the conclusion from the hypotheses, when they are conveniently reformulated in an elementary form (equivalent to the abstract form appearing in classical mathematics).

3. Dynamical Method in Constructive Algebra

In constructive analysis à la Bishop, or in certified numerical analysis, real numbers and continuous real functions are considered as given by finite rational approximations.

This corresponds to the Poincaré’s program: “Ne jamais perdre de vue que toute proposition sur l’infini doit être traduction, l’énoncé abrégé de propositions sur le fini.” (Never lose sight of the fact that every proposition concerning
infinity must be the translation, the precise statement of propositions concerning
the finite.)

Abstract classical algebra uses also many kinds of infinite objects, but in
general without finite approximations. Also, contrarily to the infinite objects of
Analysis, these algebraic objects are often shown lacking completely of existence.
E.g. the algebraic closure of a general field cannot be “constructed”, contrarily
to the completion of a general metric space.

The dynamical method considers that algebraic objects that are too abstract
(too abstract, because they are impossible to construct) can nevertheless be
considered through their finite approximations. Approximations of what? Ap-
proximations of more accurate compatible finite approximations! So, existence
is replaced by formal compatibility, in the spirit of Hilbert’s program.

E.g., the elementary definition of Krull dimension was obtained by consid-
ering finite approximations of chains of prime ideals.

In the talk we give some examples of this dynamical machinery: deciphering
abstract proofs that use abstract infinite objects, and transforming these proofs
in concrete algorithms dealing with finite approximations or the ideal objects.

We think that this is a very general method and that we have in hands
the tools for realizing the Poincaré’s program (and the Hilbert’s program) for
abstract algebra.

This corresponds to a logical framework called “Geometric theories”.
