Geometric theories for real number algebra without sign test or dependent choice axiom

CCC2024 Nice 02-04 oct

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My favorite quotes

Henri Poincaré, in *La logique de l'infini* (Revue de Métaphysique et de Morale, 1909).
As for me, I would propose that we be guided by the following rules:

1. Never consider any objects but those capable of being defined in a finite number of words;
2. Never lose sight of the fact that every proposition concerning infinity must be the translation, the precise statement of propositions concerning the finite;
3. Avoid nonpredicative classifications and definitions.

Karl Marx, *Comments on the latest Prussian censorship instruction*, 1843 (cited by Georges Perec in *Les Choses).*
Truth includes not only the result but also the path to it. The investigation of truth must itself be true; true investigation is developed truth, the dispersed elements of which are brought together in the result.
http://www.marxists.org/archive/marx/works/1842/02/10.htm.

Works directly related to this talk

About a new method for computing in algebraic number fields. Jean Della Dora, Claire Dicrescenzo and Dominique Duval. LNCS 204, Springer, (1985)
Dynamical method in algebra: Effective Nullstellensätze. Coste, L-, Roy. Annals of Pure and Applied Logic 111, (2001) https://arxiv.org/abs/1701.05794,
A logical approach to abstract algebra. Coquand, L-. Math. Struct. in Comput. Science 16, (2006) http://hlombardi.free.fr/publis/AlgebraLogicCoqLom.pdf
Structures algébriques dynamiques, espaces topologiques sans points et programme de Hilbert. L-. Ann. Pure Appl. Logic 137, (2006)
http://hlombardi.free.fr/publis/W2FTOP.pdf
A first proposal: Geometric theories for real number algebra without sign test or dependent choice axiom. L-, Mahboubi, https://arxiv.org/abs/2408.10290
A draft: *Théories Géométriques pour l'algèbre constructive. Part I.*
http://hlombardi.free.fr/TGM.pdf

Main questions

What are the algebraic properties of $\mathbb{R}$?

A priori we use the signature $\boxed{\Sigma_{Aso} = (\cdot = 0, \cdot \geqslant 0, \cdot > 0\,;\, \cdot + \cdot, \cdot \times \cdot, - \cdot, 0, 1)}$
Artin-Tarski formalisation works only for discrete ordered fields, not for $\mathbb{R}$.

What is a real closed field?

What is an o-minimal structure over a real closed field?

Constructively!

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Finitary dynamical theories

A purely computational version of finitary geometric theories. These *coherent* theories are associated to coherent Grothendieck toposes.
As in natural deduction, axioms and theorems are seen as *deduction rules*. Proofs are replaced by computation trees. This is a kind of natural deduction for the poor man: no implication connector, no negation connector, no universal quantifier.
No conflict concerning the intuitive meaning computations.
Thus no conflict between classical and intuitionist logic.
Adding classical logic produces a conservative extension. Skolemization is also conservative.
Language and axioms are understood as "sets". Possible conflict here when the external mathematical word is necessary. E.g. the theory of discrete real closed fields with $\mathbb{R}$ as set of constants.
No conflict if the language is countable and axioms are decidable.

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Infinitary dynamical theories

Computational version of geometric theories.
Use infinite disjunctions.
Does adding classical logic produce a conservative extension? (Barr's theorem)
External proofs concerning these disjunctions are in the intuitive mathematical world.
From a computational point view, it seems unavoidable to use an external constructive mathematical world.

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Problems when formalising algebraic properties of $\mathbb{R}$

Discreteness axioms

Ed$_{\neq}$ $\vdash x = 0$ **or** $x \neq 0$ **OT** $\vdash x \geqslant 0$ **or** $x \leqslant 0$

Intermediate value theorem

RCF$_n$ $a < b, P(a)P(b) < 0 \vdash \exists x \big(P(x) = 0,\, a < x < b\big)$ $\deg(P) \leqslant n$

Heyting axiom for ordered field

HOF $(x > 0 \Rightarrow 1 = 0) \Rightarrow x \leqslant 0$ this axiom is not geometric

Ed$_{\neq}$ and **OT** are not valid for $\mathbb{R}$. One has instead

OTF $x + y > 0 \vdash x > 0$ **or** $y > 0$

OTF$^{\times}$ $xy < 0 \vdash x < 0$ **or** $y < 0$

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Rational maps well defined in $\mathbb{R}$

When you drop **Ed**$_{\neq}$ and **OT** and use instead **OTF** and **HOF**, the formal intuitionist theory does not prove the existence of $\sup(x, y)$. But this function is well defined over $\mathbb{R}$.
In order to describe a constructive notion of a *non* discrete ordered field in a dynamical theory we need to enlarge the signature.

Signature : $\boxed{\Sigma_{Co} = (\cdot = 0, \cdot \geqslant 0, \cdot > 0\,;\, \cdot + \cdot, \cdot \times \cdot, \cdot \vee \cdot, - \cdot, \mathrm{Fr}(\cdot,\cdot), 0, 1)}$

We dont use **HOF**, we add convenient axioms. E.g., the ring is local. This is not completely satisfactory. A better version than introducing Fr with its axioms should be to add all rational semialgebraic continuous functions defined over $\mathbb{Q}$, with convenient axioms. But it is too cumbersome.
The moto is: everything acceptable for $\mathbb{R}$ should be already defined in the discrete case.

Square roots. Euclidean fields

Il the case of discrete fields, an euclidean field is simply a real field where squares are fourth powers.
A field is real (Artin!) when -1 is not a sum of squares.
In this case the $\geqslant 0$ elements are simply the squares.
For the *non* discrete case, things are much more complicated. E.g. we don't know today how to prove that starting with a *non* discrete ordered field and adding formally positive square roots of nonnegative elements gives a local ring.

Virtual real roots

Bishop proves the validity of $\mathbf{RCF}_n$ by using dependent choice, but this is rather inelegant. Moreover it should not be a good way for introducing formally real roots of monic polynomials.
In the discrete case one uses Thom's coding of real roots in order to describe precisely the real closure of discrete ordered fields.
Virtual real roots have been invented in order to provide continuous semialgebraic functions of the coefficients that cover all the real roots. See: Laureano González-Vega, Henri Lombardi and Louis Mahé. *Virtual roots of real polynomials*. J. Pure Appl. Algebra **124** (1998), pp. 147–166. http://arxiv.org/abs/1712.01952.
The Budan-Fourier method counts exactly the virtual roots!

Real closed rings *non* discrete real closed fields

f-rings with virtual roots can be described by a purely equational theory.
They were invented by Niels Schwartz in 1984 under the name of *real closed rings* (without using virtual roots) in a framework of highly nonconstructive classical mathematics.
A rather good definition of a *non* discrete real closed field should be a *local real closed ring*

Towards a dynamical theory of o-minimal structures

Thanks for your attention