Constructive Krull Dimension. I: Integral Extensions.

Thierry Coquand (*) Lionel Ducos (†) Henri Lombardi (‡), Claude Quitté (§)

May, 2006

Abstract

We give a constructive approach to the well known classical theorem saying that an integral extension doesn't change the Krull dimension.

MSC 2000: 13C15, 03F65, 13A15, 13E05

Introduction

In this paper and the following one (Constructive Krull Dimension. II: Noetherian Rings) we investigate some classical topics about Krull dimension from a constructive point of view.

The fact that a constructive theory of Krull dimension, avoiding Choice and Third Excluded Middle Principle is indeed possible was made clear by the work of Joyal and Español ([Joy, Esp, Esp2] 1975, 1982, 1986).

Concrete applications appeared when more easily manageable characterizations of Krull dimension appeared ([Lom, CL] 2002, [CLR, CLQ2, CL2] 2005). Notice that a similar elementary definition follows easily from a result of Brenner ([Bre] 2003).

Some celebrated theorems of commutative algebra as the Serre's splitting off, the Bass stable range theorem, the Bass cancellation theorem, the Forster-Swan theorem, the Brewer-Costa-Maroscia theorem and the Eisenbud-Evans-Storch theorem, have now a completely algorithmic version (see [Coq, CLQ, Duc, CLS, LQY]). For the Serre's splitting off and the Forster-Swan theorem the constructive approach has eventually lead to stronger versions than the previously existing ones, giving a positive answer to a question of Heitmann in the memorable non-noetherian paper [Hei].

In this paper we give a constructive approach to the well known classical theorem saying that an integral extension doesn't change the Krull dimension.

So we get that for such an extension $\mathbf{A} \subset \mathbf{B}$ we have an algebraic machinery that transforms the production of identities whose meaning is $\mathsf{Kdim}\,\mathbf{A} \leq n$ in the production of identities whose meaning is $\mathsf{Kdim}\,\mathbf{B} \leq n$, and vice-versa.

The paper is written in the usual style of constructive algebra, with [MRR] as a basic reference.

---

* Chalmers, University of Göteborg, Sweden, email: coquand@cs.chalmers.se

† Laboratoire de Mathématiques, SP2MI, Boulevard 3, Teleport 2, BP 179, 86960 FUTUROSCOPE Cedex, FRANCE, email: ducos@mathlabo.univ-poitiers.fr

‡ Equipe de Mathématiques, CNRS UMR 6623, UFR des Sciences et Techniques, Université de Franche-Comté, 25 030 BESANCON cedex, FRANCE, email: henri.lombardi@univ-fcomte.fr

§ Laboratoire de Mathématiques, SP2MI, Boulevard 3, Teleport 2, BP 179, 86960 FUTUROSCOPE Cedex, FRANCE, email: quitte@mathlabo.univ-poitiers.fr

1 Constructive Krull dimension

In this section we recall some elementary characterizations of the Krull dimension. Proofs may be found e.g. in [CLR, CLQ2].

Let us consider a commutative ring $\mathbf{A}$.

A *filter* is a saturated multiplicative monoid in $\mathbf{A}$. A *prime filter* is a filter not equal to $\mathbf{A}$ whose corresponding localization gives a local ring. Within classical mathematics, prime filters are exactly the complements of prime ideals, a *maximal filter* is prime and its complement is a minimal prime.

The radical of an ideal $\mathfrak{a}$ will be noted as $\mathrm{D}_{\mathbf{A}}(\mathfrak{a})$. We write $\mathrm{D}_{\mathbf{A}}(x_1,\ldots,x_m)$ for $\mathrm{D}_{\mathbf{A}}(\langle x_1,\ldots,x_m\rangle)$. The *Zariski (distributive) lattice* $\mathsf{Zar}\,\mathbf{A}$ is defined as the set

$$\mathsf{Zar}\,\mathbf{A} = \{\mathrm{D}_{\mathbf{A}}(x_1,\ldots,x_m)\,|\,m\in\mathbb{N},\,x_1,\ldots,x_m\in\mathbf{A}\}$$

ordered by inclusion. Within classical mathematics, $\mathsf{Zar}\,\mathbf{A}$ is isomorphic to the lattice of compact open subsets of the Zariski spectrum $\mathsf{Spec}\,\mathbf{A}$. The isomorphism is given by

$$\mathrm{D}_{\mathbf{A}}(x_1,\ldots,x_m) \longmapsto \mathfrak{D}_{\mathbf{A}}(x_1)\cup\cdots\cup\mathfrak{D}_{\mathbf{A}}(x_m)$$

where $\mathfrak{D}_A(a) = \{\,\mathfrak{p}\in\mathsf{Spec}\,\mathbf{A}\,|\,a\notin\mathfrak{p}\,\}$.

1.1 Krull boundaries

Recall that for an ideal $\mathfrak{a}$ and an element a of $\mathbf{A}$ we have the notation

$$(\mathfrak{a}:a^\infty)_{\mathbf{A}} = \bigcup_{n\in\mathbb{N}}(\mathfrak{a}:a^n)_{\mathbf{A}} = \{y\in\mathbf{A}\mid \exists\, n\in\mathbb{N},\ ya^n\in\mathfrak{a}\}$$

Definition 1.1 *Let $x_0,\ldots,x_\ell$ be a sequence of elements of a commutative ring $\mathbf{A}$.*

1. *We define inductively an* iterated boundary monoid *$\mathcal{S}_{\mathbf{A}}(x_0,\ldots,x_\ell)$ for this sequence by:*

$$\mathcal{S}_{\mathbf{A}}() = \{1\} \qquad \textit{and} \qquad \mathcal{S}_{\mathbf{A}}(x_0,\ldots,x_\ell) = x_0^{\mathbb{N}}\left(\mathcal{S}_{\mathbf{A}}(x_1,\ldots,x_\ell)+\mathbf{A}x_0\right). \tag{1}$$

   E.g.,

$$\mathcal{S}_{\mathbf{A}}(x_0,x_1,x_2) = x_0^{\mathbb{N}}\Big(x_1^{\mathbb{N}}\left(x_2^{\mathbb{N}}(1+\mathbf{A}x_2)+\mathbf{A}x_1\right)+\mathbf{A}x_0\Big).$$

2. *We define inductively an* iterated boundary ideal *$\mathcal{N}_{\mathbf{A}}(x_0,\ldots,x_\ell)$ for this sequence by:*

$$\mathcal{N}_{\mathbf{A}}() = \{0\} \qquad \textit{and} \qquad \mathcal{N}_{\mathbf{A}}(x_0,\ldots,x_\ell) = (\mathcal{N}_{\mathbf{A}}(x_0,\ldots,x_{\ell-1}):x_\ell^\infty)_{\mathbf{A}}+\mathbf{A}x_\ell. \tag{2}$$

   E.g.,

$$\mathcal{N}_{\mathbf{A}}(x_0) = (0:x_0^\infty)+\mathbf{A}x_0, \qquad \mathcal{N}_{\mathbf{A}}(x_0,x_1) = \Big(((0:x_0^\infty)+\mathbf{A}x_0):x_1^\infty\Big)+\mathbf{A}x_1.$$

For any $a\in\mathbf{A}$, $\mathcal{N}_{\mathbf{A}}(a)$ meets any maximal ideal and $\mathcal{S}_{\mathbf{A}}(a)$ meets any maximal filter.

The inductive definition of $\mathcal{S}(x_0,\ldots,x_d)$ can be understood with the constructor $M_a := S\mapsto a^{\mathbb{N}}(S+\mathbf{A}a)$ where $a\in\mathbf{A}$ and S is an arbitrary multiplicative monoid in $\mathbf{A}$. More precisely

$$\mathcal{S}(x_0,\ldots,x_d) = M_{x_0}\circ M_{x_1}\circ\cdots\circ M_{x_d}(\{1\})$$

Similarly the inductive definition of $\mathcal{N}(x_0, \ldots, x_d)$ can be understood with the "dual" constructor $I_a := \mathfrak{a} \mapsto (\mathfrak{a} : a^\infty)_{\mathbf{A}} + \mathbf{A}a$ where $\mathfrak{a}$ is an arbitrary ideal of $\mathbf{A}$. More precisely

$$\mathcal{N}(x_0, \ldots, x_d) = I_{x_d} \circ \cdots \circ I_{x_1} \circ I_{x_0}(\{0\})$$

We have the equivalences

$$0 \in \mathcal{S}_{\mathbf{A}}(x_0, \ldots, x_d) \;\Leftrightarrow\; 1 \in \mathcal{N}_{\mathbf{A}}(x_0, \ldots, x_d) \;\Leftrightarrow\; \mathcal{N}_{\mathbf{A}}(x_0, \ldots, x_{i-1}) \cap \mathcal{S}_{\mathbf{A}}(x_i, \ldots, x_d) \neq \emptyset \tag{3}$$

This justifies reversing the order between $M_{x_0} \circ M_{x_1} \circ \cdots \circ M_{x_d}$ and $I_{x_d} \circ \cdots \circ I_{x_1} \circ I_{x_0}$. When $0 \in \mathcal{S}_{\mathbf{A}}(x_0, \ldots, x_d)$ we will say that the sequence $x_0, \ldots, x_d$ is *pseudo singular*.

Remark. In [CLR], where the Krull boundaries are defined for the first time, the boundary ideal uses a slightly different constructor $\mathfrak{a} \mapsto (\sqrt{\mathfrak{a}} : \mathbf{A}a)_{\mathbf{A}} + \mathbf{A}a$. Let us denote by $\mathfrak{V}_{\mathbf{A}}(\mathfrak{a})$ the closed subset of $\mathsf{Spec}\,\mathbf{A}$ defined by $\mathfrak{a}$ (i.e., the complement of $\mathfrak{D}_{\mathbf{A}}$). The name "boundary of $\mathfrak{a}$" for $\mathfrak{b} = (\mathrm{D}_{\mathbf{A}}(0) : \mathfrak{a}) + \mathfrak{a}$ comes form the fact that $\mathfrak{V}_{\mathbf{A}}(\mathfrak{b}) = \overline{\mathfrak{D}_{\mathbf{A}}(\mathfrak{a})} \cap \mathfrak{V}_{\mathbf{A}}(\mathfrak{a})$ is the boundary of $\mathfrak{V}_{\mathbf{A}}(\mathfrak{a})$ inside $\mathsf{Spec}\,\mathbf{A}$ (in classical mathematics). This gives an intuitive explanation for the fact that the dimension on $\mathbf{A}/\mathfrak{b}$ is stricty lesser than the dimension of $\mathbf{A}$: the boundary of any subvariety in a variety is always strictly lesser than the variety itself.

Next, Fred Richman defined in [Ric] another boundary with the constructor we use here.

In fact the boundary ideal $(\sqrt{0} : \mathbf{A}a)_{\mathbf{A}} + \mathbf{A}a$ of [CLR] contains the Richman boundary ideal $\mathcal{N}_{\mathbf{A}}(a)$ and they have the same radical. So the two quotient rings have isomorphic Zariski lattices, and the difference is not really important.

Perhaps the most intrinsic definitions would be to consider $\sqrt{\mathcal{N}_{\mathbf{A}}(x_0, \ldots, x_n)}$ and the saturation of the monoid $\mathcal{S}_{\mathbf{A}}(x_0, \ldots, x_n)$.

1.2 Characterizations of Krull dimension

First recall that a ring has Krull dimension -1 if and only if it is trivial.

Theorem 1.2 *Let* $\mathbf{A}$ *be a commutative ring and* $d \in \mathbb{N}$. *The following are equivalent:*

1. *(classical definition) Any increasing chain of primes has length* $\leq d$ *(i.e., the number of primes in the chain is* $\leq d+1$*). The maximal length of such a chain belongs to* $\mathbb{N} \cup \{\infty\}$ *and is denoted by* $\mathsf{Kdim}\,\mathbf{A} \leq d$.
2. *(induction using ideal boundary) For any* $a \in \mathbf{A}$, $\mathsf{Kdim}\,(\mathbf{A}/\mathcal{N}_{\mathbf{A}}(a)) \leq d-1$.
3. *(induction using monoid boundary) For any* $a \in \mathbf{A}$, $\mathsf{Kdim}\,(\mathcal{S}_{\mathbf{A}}(a)^{-1}\mathbf{A}) \leq d-1$.
4. *(iterated boundaries) Any sequence* $x_0, \ldots, x_d$ *in* $\mathbf{A}$ *is pseudo singular.*
5. *(symmetric form) For any* $x_0, \ldots, x_d \in \mathbf{A}$, *there exist* $a_0, \ldots, a_d \in \mathbf{A}$ *such that*

$$\left\{\begin{array}{rcl} a_0 x_0 & \in & \mathrm{D}_{\mathbf{A}}(0) \\ a_1 x_1 & \in & \mathrm{D}_{\mathbf{A}}(a_0, x_0) \\ a_2 x_2 & \in & \mathrm{D}_{\mathbf{A}}(a_1, x_1) \\ & \vdots & \\ a_d x_d & \in & \mathrm{D}_{\mathbf{A}}(a_{d-1}, x_{d-1}) \\ 1 & \in & \mathrm{D}_{\mathbf{A}}(a_d, x_d) \end{array}\right. \tag{4}$$

Moreover points 2, 3, 4, 5 lead to constructively equivalent definitions of the Krull dimension.

Two sequences $a_0, \ldots, a_d$ and $x_0, \ldots, x_d$ satisfying the point 5 will be called *complementary sequences*. In the lattice notation this gives

$$\left\{ \begin{array}{rcl} \mathrm{D}_{\mathbf{A}}(a_0) \wedge \mathrm{D}_{\mathbf{A}}(x_0) & \leq & 0_{\mathsf{Zar}\,\mathbf{A}} \\ \mathrm{D}_{\mathbf{A}}(a_1) \wedge \mathrm{D}_{\mathbf{A}}(x_1) & \leq & \mathrm{D}_{\mathbf{A}}(a_0) \vee \mathrm{D}_{\mathbf{A}}(x_0) \\ \mathrm{D}_{\mathbf{A}}(a_2) \wedge \mathrm{D}_{\mathbf{A}}(x_2) & \leq & \mathrm{D}_{\mathbf{A}}(a_1) \vee \mathrm{D}_{\mathbf{A}}(x_1) \\ & \vdots & \\ \mathrm{D}_{\mathbf{A}}(a_d) \wedge \mathrm{D}_{\mathbf{A}}(x_d) & \leq & \mathrm{D}_{\mathbf{A}}(a_{d-1}) \vee \mathrm{D}_{\mathbf{A}}(x_{d-1}) \\ 1_{\mathsf{Zar}\,\mathbf{A}} & \leq & \mathrm{D}_{\mathbf{A}}(a_d) \vee \mathrm{D}_{\mathbf{A}}(x_d) \end{array} \right.$$

Here are light variations on the formulations for the point 4.

1. For any $x_0, \ldots, x_d$ there exist $a_0, \ldots, a_d \in \mathbf{A}$ and $m_0, \ldots, m_d \in \mathbb{N}$ such that

$$x_0^{m_0}(\cdots(x_d^{m_d}(1 + a_d x_d) + \cdots) + a_0 x_0) = 0$$

2. $0 \in \mathcal{S}(x_0, \ldots, x_d) = x_0^{\mathbb{N}}(x_1^{\mathbb{N}}(\cdots(x_d^{\mathbb{N}}(1 + \mathbf{A}x_d) \cdots + \mathbf{A}x_1) + \mathbf{A}x_0)$
3. $1 \in \mathcal{N}(x_0, \ldots, x_d) = ((\cdots(0 : x_0^{\infty}) + \mathbf{A}x_0 \cdots) : x_d^{\infty}) + \mathbf{A}x_d$
4. For any $x_0, \ldots, x_d \in \mathbf{A}$, there exist $n \in \mathbb{N}$ such that:

$$(x_0 \cdots x_d)^n \in \mathbf{A}(x_0 \cdots x_{d-1})^n x_d^{n+1} + \mathbf{A}(x_0 \cdots x_{d-2})^n x_{d-1}^{n+1} + \cdots + \mathbf{A}x_0^{n+1}$$

Remarks.

In constructive mathematics the sentence “ $\mathsf{Kdim}\,\mathbf{A} \leq \ell$ ” is well defined but $\mathsf{Kdim}\,\mathbf{A}$ is not in general a well defined element of $\mathbb{N} \cup \{\infty\}$. It is remarkable that most classical theorems using Krull dimension may be put under the form “If $\mathsf{Kdim}\,\mathbf{A} \leq \ell$ then …”.

The basic fact that $\mathsf{Kdim}\,\mathbf{K}[X_1, \ldots, X_\ell] = \ell$ when $\mathbf{K}$ is a discrete field has a very simple proof, see [CL2]. As a consequence, the usual “geometrical rings” have a well defined Krull dimension in constructive mathematics. This means e.g., that the construction of complementary sequences is given by an effective procedure in such rings. See also [Lom] for an explicit generalization of the Nullstellensatz.

1.3 Some basic facts

Following simple facts show a kind of strong “ duality ” between addition and multiplication, ideals and filters, and the two kinds of Krull boundaries.

Fact 1.3 (Krull boundaries, localizations and quotients)
Let $x_0, \ldots, x_\ell \in \mathbf{A}$, S a monoid and $\mathfrak{a}$ an ideal. One has:

1. $\mathcal{S}_{\mathbf{A}/\mathfrak{a}}(x_0, \ldots, x_\ell) = \mathcal{S}_{\mathbf{A}}(x_0, \ldots, x_\ell) \mod \mathfrak{a}$.
2. $\mathcal{N}_{S^{-1}\mathbf{A}}(x_0, \ldots, x_\ell) = S^{-1}\mathcal{N}_{\mathbf{A}}(x_0, \ldots, x_\ell)$.
3. (a) $\mathcal{S}_{S^{-1}\mathbf{A}}(x_0, \ldots, x_\ell) = S^{-1}\Big(x_0^{\mathbb{N}}(x_1^{\mathbb{N}}(\cdots(x_d^{\mathbb{N}}(S + \mathbf{A}x_d) \cdots + \mathbf{A}x_1) + \mathbf{A}x_0)\Big)$.
   (b) *If $S = \mathcal{S}(x_\ell)$ then $\mathcal{S}_{S^{-1}\mathbf{A}}(x_0, \ldots, x_{\ell-1}) = S^{-1}\mathcal{S}_{\mathbf{A}}(x_0, \ldots, x_\ell)$.*
4. (a) $\mathcal{N}_{\mathbf{A}/\mathfrak{a}}(x_0, \ldots, x_\ell) = \Big(((\cdots(\mathfrak{a} : x_0^{\infty}) + \mathbf{A}x_0 \cdots) : x_d^{\infty}) + \mathbf{A}x_d\Big)/\mathfrak{a}$.

(b) If $\mathfrak{a} = \mathcal{N}(x_0)$ *then* $\mathcal{N}_{\mathbf{A}/\mathfrak{a}}(x_1, \ldots, x_\ell) = \mathcal{N}_{\mathbf{A}}(x_0, \ldots, x_\ell)/\mathfrak{a}$.

The fact that Krull dimension cannot increase by localization or quotient is direct and constructive from the constructive definition. Some converse implications are given in the following lemmas.

Lemma 1.4 (Krull dimension and quotients)
Let $\mathfrak{a}, \mathfrak{b}$ *be ideals of* $\mathbf{A}$. *Then*

$$\mathsf{Kdim}\,\mathbf{A}/(\mathfrak{a} \cap \mathfrak{b}) = \mathsf{Kdim}\,\mathbf{A}/\mathfrak{a}\mathfrak{b} = \max\{\mathsf{Kdim}\,\mathbf{A}/\mathfrak{a}, \mathsf{Kdim}\,\mathbf{A}/\mathfrak{b}\}$$

This remains true for finite intersections and products of ideals.

Proof. The first equality comes from $\mathrm{D}_{\mathbf{A}}(\mathfrak{a} \cap \mathfrak{b}) = \mathrm{D}_{\mathbf{A}}(\mathfrak{a}\mathfrak{b})$.
If $x_0, \ldots, x_n \in \mathbf{A}$, $0 \in \mathcal{S}_{\mathbf{A}/\mathfrak{a}}(x_0, \ldots, x_n)$ means that $\mathcal{S}_{\mathbf{A}}(x_0, \ldots, x_n)$ meets $\mathfrak{a}$. So $\mathsf{Kdim}\,\mathbf{A}/\mathfrak{a} \leq n$ and $\mathsf{Kdim}\,\mathbf{A}/\mathfrak{b} \leq n$ means that $\mathcal{S}_{\mathbf{A}}(x_0, \ldots, x_n)$ meets $\mathfrak{a}$ and $\mathfrak{b}$ for any $x_0, \ldots, x_n \in \mathbf{A}$, which is equivalent to $\mathcal{S}_{\mathbf{A}}(x_0, \ldots, x_n)$ meets the product $\mathfrak{a}\,\mathfrak{b}$. □

Lemma 1.5 (Krull dimension and localizations)
Let $S_1, \ldots, S_\ell$ *be* comaximal monoids *in* $\mathbf{A}$, *i.e., any ideal meeting all the* S_i *equals* $\mathbf{A}$. *Then*

$$\mathsf{Kdim}\,\mathbf{A} = \max\left\{\mathsf{Kdim}\,S_i^{-1}\mathbf{A} \mid i = 1, \ldots, \ell\right\}$$

Proof. Straightforward and constructive. □

2 Integral extensions

Let $\mathbf{A} \subset \mathbf{B}$ be an integral extension. Within classical mathematics the Lying Over is equivalent to the following inclusion for any ideal $\mathfrak{a}$:

$$\mathbf{A} \cap \mathfrak{a}\mathbf{B} \subseteq \mathrm{D}_{\mathbf{A}}(\mathfrak{a}).$$

The following classical lemma gives a slightly more precise result, without using prime ideals. It is easily proven with a determinant trick.

Lemma 2.1 *Let* $\mathbf{A} \subseteq \mathbf{B}$ *be an integral extension and an ideal* $\mathfrak{a} \subseteq \mathbf{A}$. *Then any* $b \in \mathfrak{a}\mathbf{B}$ *is integral over the ideal* $\mathfrak{a}$, *i.e.,*

$$\exists\ a_1 \in \mathfrak{a},\ a_2 \in \mathfrak{a}^2, \ldots, a_n \in \mathfrak{a}^n, \qquad b^n + a_1 b^{n-1} + \cdots + a_{n-1}b + a_n = 0$$

As a consequence we get

$$\mathbf{A} \cap \mathfrak{a}\mathbf{B} \subseteq \mathrm{D}_{\mathbf{A}}(\mathfrak{a}), \qquad 1 + \mathfrak{a}\mathbf{B} \subseteq (1 + \mathfrak{a})^{\mathrm{sat}_{\mathbf{B}}}.$$

We give now a slight generalization.

Lemma 2.2 *Let* $\mathbf{A} \subseteq \mathbf{B}$ *be an integral extension with an ideal* $\mathfrak{a} \subseteq \mathbf{A}$, *an ideal* $\mathfrak{b} \subseteq \mathbf{B}$, *and a monoid* $S \subseteq \mathbf{A}$. *Then one has*

$$\mathbf{A} \cap (\mathfrak{b} + \mathfrak{a}\mathbf{B}) \subseteq \mathrm{D}_{\mathbf{A}}((\mathbf{A} \cap \mathfrak{b}) + \mathfrak{a}), \qquad S + \mathfrak{a}\mathbf{B} \subseteq (S + \mathfrak{a})^{\mathrm{sat}_{\mathbf{B}}}.$$

Proof. Let $a \in \mathbf{A} \cap (\mathfrak{b} + \mathfrak{a}\mathbf{B})$. Using Lemma 2.1 in the integral extension $\mathbf{B}/\mathfrak{b}$ of $\mathbf{A}/(\mathbf{A} \cap \mathfrak{b})$, we deduce that $a^{\mathbb{N}}$ meets $\mathfrak{a} + \mathfrak{b}$. But $a \in \mathbf{A}$ and $\mathfrak{a} \subseteq \mathbf{A}$, so $a^{\mathbb{N}}$ meets $\mathfrak{a} + (\mathfrak{b} \cap \mathbf{A})$.
Let $s \in S + \mathfrak{a}\mathbf{B}$. We use Lemma 2.1 in the integral extension $S^{-1}\mathbf{B}$ of $S^{-1}\mathbf{A}$. Since s belongs to $(1 + \mathfrak{a}S^{-1}\mathbf{B})^{\mathrm{sat}_\mathbf{B}}$, it belongs also to $(1 + \mathfrak{a}S^{-1}\mathbf{A})^{\mathrm{sat}_\mathbf{B}}$, and this implies in $\mathbf{B}$ that s belongs to $(S + \mathfrak{a})^{\mathrm{sat}_\mathbf{B}}$. □

Proposition 2.3 *Let $\mathbf{A} \subseteq \mathbf{B}$ be an integral extension and $a_0, \ldots, a_d \in \mathbf{A}$. Then*

$$\mathbf{A} \cap \mathcal{N}_\mathbf{B}(a_0, \ldots, a_d) \subseteq \mathrm{D}_\mathbf{A}(\mathcal{N}_\mathbf{A}(a_0, \ldots, a_d)), \qquad \mathcal{S}_\mathbf{B}(a_0, \ldots, a_d) \subseteq \mathcal{S}_\mathbf{A}(a_0, \ldots, a_d)^{\mathrm{sat}_\mathbf{B}}.$$

Proof by induction on d.
First,

$$\begin{array}{rcll} \mathbf{A} \cap \mathcal{N}_\mathbf{B}(a_0, \ldots, a_d) & = & \mathbf{A} \cap ((\mathcal{N}_\mathbf{B}(a_0, \ldots, a_{d-1}) : a_d^\infty)_\mathbf{B} + \mathbf{B}a_d) & \text{(definition)} \\ & \subseteq & \mathrm{D}_\mathbf{A}(\mathbf{A} \cap (\mathcal{N}_\mathbf{B}(a_0, \ldots, a_{d-1}) : a_d^\infty)_\mathbf{B} + \mathbf{A}a_d) & \text{(Lemma 2.2)} \\ & = & \mathrm{D}_\mathbf{A}\left(((\mathbf{A} \cap \mathcal{N}_\mathbf{B}(a_0, \ldots, a_{d-1})) : a_d^\infty)_\mathbf{A} + \mathbf{A}a_d\right) & \\ & \subseteq & \mathrm{D}_\mathbf{A}\left((\mathrm{D}_\mathbf{A}(\mathcal{N}_\mathbf{A}(a_0, \ldots, a_{d-1})) : a_d^\infty)_\mathbf{A} + \mathbf{A}a_d\right) & \text{(induction)} \\ & = & \mathrm{D}_\mathbf{A}\left((\mathcal{N}_\mathbf{A}(a_0, \ldots, a_{d-1}) : a_d^\infty)_\mathbf{A} + \mathbf{A}a_d\right) & \\ & = & \mathrm{D}_\mathbf{A}(\mathcal{N}_\mathbf{A}(a_0, \ldots, a_d)). & \text{(definition)} \end{array}$$

Second,

$$\begin{array}{rcll} \mathcal{S}_\mathbf{B}(a_0, \ldots, a_d) & = & a_0^\mathbb{N}\left(\mathcal{S}_\mathbf{B}(a_1, \ldots, a_d) + \mathbf{B}a_0\right) & \text{(definition)} \\ & \subseteq & a_0^\mathbb{N}\left(\mathcal{S}_\mathbf{A}(a_1, \ldots, a_d)^{\mathrm{sat}_\mathbf{B}} + \mathbf{B}a_0\right) & \text{(induction)} \\ & \subseteq & a_0^\mathbb{N}\left(\mathcal{S}_\mathbf{A}(a_1, \ldots, a_d) + \mathbf{B}a_0\right)^{\mathrm{sat}_\mathbf{B}} & \\ & \subseteq & a_0^\mathbb{N}\left(\mathcal{S}_\mathbf{A}(a_1, \ldots, a_d) + \mathbf{A}a_0\right)^{\mathrm{sat}_\mathbf{B}} & \text{(Lemma 2.2)} \\ & \subseteq & \left(a_0^\mathbb{N}\left(\mathcal{S}_\mathbf{A}(a_1, \ldots, a_d) + \mathbf{A}a_0\right)\right)^{\mathrm{sat}_\mathbf{B}} & \\ & = & \mathcal{S}_\mathbf{A}(a_0, \ldots, a_d)^{\mathrm{sat}_\mathbf{B}}. & \text{(definition)} \end{array}$$

□

Corollary 2.4 *If $\mathbf{A} \subseteq \mathbf{B}$ is an integral extension then $\mathsf{Kdim}\,\mathbf{A} \le \mathsf{Kdim}\,\mathbf{B}$.*

The reverse inequality will be shown in a more general setting in Section 3.

Proof. The constructive meaning of $\mathsf{Kdim}\,\mathbf{A} \le \mathsf{Kdim}\,\mathbf{B}$ is the implication

$$\mathsf{Kdim}\,\mathbf{B} \le d \Longrightarrow \mathsf{Kdim}\,\mathbf{A} \le d \qquad \text{(for any } d \ge -1\text{)}.$$

The case $d = -1$ is clear. Assume $\mathsf{Kdim}\,\mathbf{B} \le d$ with $d \ge 0$. Let $a_0, \ldots, a_d \in \mathbf{A}$, we have $0 \in \mathcal{S}_\mathbf{B}(a_0, \ldots, a_d)$, so $0 \in \mathcal{S}_\mathbf{A}(a_0, \ldots, a_d)^{\mathrm{sat}_\mathbf{B}}$ and $0 \in \mathcal{S}_\mathbf{A}(a_0, \ldots, a_d)$. □

3 Algebraic extensions

Recall that elements in a ring are *comaximal* when they generate the ideal $\langle 1 \rangle$. Equivalently the monoids generated by these elements are comaximal.

Definition 3.1 *Let $\mathbf{A} \subseteq \mathbf{B}$ be an extension of commutative rings. We say that $x \in \mathbf{B}$ is* algebraic *over $\mathbf{A}$ if there exist comaximal elements $a_0, \ldots, a_k \in \mathbf{A}$ such that $\sum_i a_i x^i = 0$. We say that $\mathbf{B}$ is* algebraic *over $\mathbf{A}$ when any element of $\mathbf{B}$ is algebraic over $\mathbf{A}$.*

Remark. When $\mathbf{A}$ is a Bezout domain and $\mathbf{B}$ contains the fraction field of $\mathbf{A}$, we find the usual notion of algebraic elements in field extensions. But in general it seems that the subset of $\mathbf{B}$ made of elements that are algebraic over $\mathbf{A}$ is not necessarily a subring of $\mathbf{B}$.

Lemma 3.2 *If $\mathbf{B}$ is algebraic over $\mathbf{A}$ then any quotient $\mathbf{B}/\mathfrak{b}$ is algebraic over $\mathbf{A}/(\mathfrak{b} \cap \mathbf{A})$.*

Lemma 3.3 *Let $\mathbf{A}$ be a reduced ring. One has*

$$\mathcal{N}_{\mathbf{A}}(x) = \mathbf{A}x + \mathrm{Ann}_{\mathbf{A}}(x) = (\mathbf{A}x^{i+1} : x^i) \qquad \forall i \geq 1$$

Let $a_0, \ldots, a_k, x \in \mathbf{A}$ such that $\sum_i a_i x^i = 0$. Then

$$\mathcal{N}_{\mathbf{A}}(a_0) \cdots \mathcal{N}_{\mathbf{A}}(a_k) \subseteq \mathcal{N}_{\mathbf{A}}(x) + \prod\nolimits_{i=0}^{k} \mathrm{Ann}_{\mathbf{A}}(a_i) \subseteq \mathcal{N}_{\mathbf{A}}(x) + \mathrm{Ann}_{\mathbf{A}}(\langle a_0, \ldots, a_k \rangle)$$

In particular if the a_i's are comaximal in $\mathbf{A}$ and $x \in \mathbf{B} \supset \mathbf{A}$ (this means that x is algebraic over $\mathbf{A}$) we get $\mathcal{N}_{\mathbf{B}}(a_0) \cdots \mathcal{N}_{\mathbf{B}}(a_k) \subseteq \mathcal{N}_{\mathbf{B}}(x)$.

Remark. The last point means that the boundary of $\mathfrak{V}_{\mathbf{B}}(x)$ is contained in the union or the boundaries of the $\mathfrak{V}_{\mathbf{B}}(a_i)$'s.

Proof. We write $x^\perp$ for $\mathrm{Ann}_{\mathbf{A}}(x)$. The first point is straightforward. In light notation this gives in particular $\mathcal{N}_{\mathbf{A}}(x) = \langle x \rangle + x^\perp$.
Let us see the second point. Let $\mathfrak{a} = \mathcal{N}_{\mathbf{A}}(x) + \prod_i \mathrm{Ann}(a_i)$. We write the proof for $k = 3$ (the general case is similar), i.e.,

$$a_3 x^3 + a_2 x^2 + a_1 x + a_0 = 0 \qquad (*)$$

We have

(1)	$\langle a_0 \rangle \subseteq \mathbf{A}x \subseteq \mathcal{N}_{\mathbf{A}}(x) \subseteq \mathfrak{a}$	from $(*)$
(2)	$\langle a_1 \rangle\, a_0^\perp \subseteq (\mathbf{A}x^2 : x^1) \subseteq \mathfrak{a}$	multiplying $(*)$ by $a_0^\perp$
(3)	$\langle a_2 \rangle\, a_1^\perp\, a_0^\perp \subseteq (\mathbf{A}x^3 : x^2) \subseteq \mathfrak{a}$	multiplying $(*)$ by $a_1^\perp\, a_0^\perp$
(4)	$\langle a_3 \rangle\, a_2^\perp\, a_1^\perp\, a_0^\perp \subseteq (\mathbf{A}x^4 : x^3) \subseteq \mathfrak{a}$	multiplying $(*)$ by $a_2^\perp\, a_1^\perp\, a_0^\perp$
(5)	$a_3^\perp\, a_2^\perp\, a_1^\perp\, a_0^\perp \subseteq \mathfrak{a}$	

Thus $\langle a_0 \rangle + \langle a_1 \rangle\, a_0^\perp + \cdots + a_3^\perp\, a_2^\perp\, a_1^\perp\, a_0^\perp \subseteq \mathfrak{a}$, whence the conclusion since

$$\prod\nolimits_i (\langle a_i \rangle + a_i^\perp) \subseteq \langle a_0 \rangle + \langle a_1 \rangle\, a_0^\perp + \cdots + a_3^\perp\, a_2^\perp\, a_1^\perp\, a_0^\perp$$

(remark that the right hand side is a sum of $5 = 4 + 1$ terms and the left hand side a sum of $16 = 2^4$ terms). For the last point, since $\langle a_0, \ldots, a_k \rangle = \langle 1 \rangle$, $\mathrm{Ann}_{\mathbf{B}}(\langle a_0, \ldots, a_k \rangle) = 0$. □

Remark. If $\mathbf{A}$ is not reduced a similar proof gives $\mathcal{N}_{\mathbf{A}}(a_0) \cdots \mathcal{N}_{\mathbf{A}}(a_k) \subseteq \mathrm{D}_{\mathbf{A}}(\mathcal{N}_{\mathbf{A}}(x)) + \prod_i (0 : a_i^\infty)$.

Theorem 3.4 *If $\mathbf{B}$ is algebraic over $\mathbf{A}$ then* $\mathsf{Kdim}\,\mathbf{B} \leq \mathsf{Kdim}\,\mathbf{A}$.

Corollary 3.5 *Let $\mathbf{A} \subseteq \mathbf{B}$ be an integral extension, then* $\mathsf{Kdim}\,\mathbf{B} = \mathsf{Kdim}\,\mathbf{A}$.

Proof. Follows from proposition 2.3 and Theorem 3.4 □

Proof of Theorem 3.4.
We prove by induction on n that $\mathsf{Kdim}\,\mathbf{A} \le n$ implies $\mathsf{Kdim}\,\mathbf{B} \le n$. The case $n = -1$ is clear.
Without loss of generality we assume $\mathbf{A}$ and $\mathbf{B}$ are reduced rings.
Assume that $\mathsf{Kdim}\,\mathbf{A} \le n$ with $n \ge 0$. For any $x \in \mathbf{B}$, we have to prove $\mathsf{Kdim}\,\mathbf{B}/\mathcal{N}_{\mathbf{B}}(x) \le n-1$.
As x is algebraic over $\mathbf{A}$, there exist comaximal elements a_i of $\mathbf{A}$ such that $\sum_{i=0}^{\ell} a_i x^i = 0$.
Lemma 3.3 gives $\prod_i \mathcal{N}_{\mathbf{B}}(a_i) \subseteq \mathcal{N}_{\mathbf{B}}(x)$. Thus Lemma 1.4 gives

$$\mathsf{Kdim}\,\mathbf{B}/\mathcal{N}_{\mathbf{B}}(x) \le \mathsf{Kdim}\left(\mathbf{B}\Big/\prod\nolimits_i \mathcal{N}_{\mathbf{B}}(a_i)\right) = \max_i\ \mathsf{Kdim}\,\mathbf{B}/\mathcal{N}_{\mathbf{B}}(a_i).$$

Induction hypothesis applies to $\mathbf{B}_i = \mathbf{B}/\mathcal{N}_{\mathbf{B}}(a_i)$ and $\mathbf{A}_i = \mathbf{A}/(\mathcal{N}_{\mathbf{B}}(a_i) \cap \mathbf{A})$. Moreover $\mathbf{A}_i$ is a quotient of $\mathbf{A}'_i = \mathbf{A}/\mathcal{N}_{\mathbf{A}}(a_i)$, so $\mathsf{Kdim}\,\mathbf{A}_i \le \mathsf{Kdim}\,\mathbf{A}'_i$ and

$$\mathsf{Kdim}\,\mathbf{B}_i \le \mathsf{Kdim}\,\mathbf{A}_i \le \mathsf{Kdim}\,\mathbf{A}'_i \le n-1.$$

□

Remark. Let $\mathbf{K}$ be the fraction field of a principal ideal domain $\mathbf{A}$. Then $\mathbf{K}$ is algebraic over $\mathbf{A}$ and $0 = \mathsf{Kdim}\,\mathbf{K} < \mathsf{Kdim}\,\mathbf{A} = 1$ if $\mathbf{A} \ne \mathbf{K}$.

References

[Bre] Brenner H. *Lifting chains of prime ideals.* J. Pure Appl. Algebra, **179** (2003), 1–5. 1

[Coq] Coquand T. *Sur un théorème de Kronecker concernant les variétés algébriques* C. R. Acad. Sci. Paris, Ser. I **338** (2004), 291–294. 1

[CL] Coquand T., Lombardi H. *Hidden constructions in abstract algebra (3) Krull dimension of distributive lattices and commutative rings*, in: Commutative ring theory and applications. Eds: Fontana M., Kabbaj S.-E., Wiegand S. Lecture notes in pure and applied mathematics vol 231. M. Dekker. (2002), 477–499. 1

[CL2] Coquand T., Lombardi H. *A short proof for the Krull dimension of a polynomial ring.* American Math. Monthly. **112** (2005), no. 9, 826–829. 1, 4

[CLQ] Coquand T., Lombardi H., Quitté C. *Generating non-noetherian modules constructively.* Manuscripta mathematica. **115** (2004), 513–520. 1

[CLQ2] Coquand T., Lombardi H., Quitté C. *Dimension de Heitmann des treillis distributifs et des anneaux commutatifs.* Preprint 2005. 1, 2

[CLR] Coquand T., Lombardi H., Roy M.-F. *An elementary characterisation of Krull dimension.* in: From Sets and Types to Analysis and Topology: Towards Practicable Foundations for Constructive Mathematics (L. Crosilla, P. Schuster, eds.) Oxford University Press. (2005) 239–244. 1, 2, 3

[CLS] Coquand T., Lombardi H., Schuster P. *A nilregular element property.* Archiv der Mathematik **85** (2005), 49–54. 1

[Duc] Ducos L. *Vecteurs unimodulaires et systèmes générateurs.* Journal of Algebra. **297** (2006), 566–583. 1

[Esp] Español L. *Constructive Krull dimension of lattices.* Rev. Acad. Cienc. Zaragoza (2). **37** (1982), 5–9. 1

[Esp2] Español L. *Dimension of boolean valued lattices and rings.* J. Pure Appl. Algebra. **42** 3 (1986), 223–236. 1

[Joy] JOYAL A. *Le théorème de Chevalley-Tarski.* Cahiers de Topologie et Géometrie Differentielle, (1975). 1

[Hei] HEITMANN, R. *Generating non-Nœtherian modules efficiently.* Michigan Math. **31** 2 (1984) 167–180. 1

[Lom] LOMBARDI H. *Dimension de Krull, Nullstellensätze et Évaluation dynamique.* Math. Zeitschrift. **242** (2002), 23–46. 1, 4

[LQY] LOMBARDI H., QUITTÉ C., YENGUI I. *Hidden constructions in abstract algebra (6) The theorem of Maroscia, Brewer and Costa.* Preprint 2005. 1

[MRR] MINES R., RICHMAN F., RUITENBURG W. *A Course in Constructive Algebra.* Springer-Verlag. Universitext (1988). 1

[Ric] RICHMAN F. *A colon approach to Krull dimension.* Manuscript. December 2004. 3

Contents