An elementary recursive bound for effective Positivstellensatz and Hilbert 17th problem

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Colloquium

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Text of the slides:

http://hlombardi.free.fr/publis/Darmstadt2017-Doc.pdf

Hilbert, 1900

http://hlombardi.free.fr/Hilbert-Mathematical_problems.pdf.
Hilbert's seventeenth problem is one of the 23 Hilbert problems set out in a celebrated list compiled in 1900 by David Hilbert. It concerns the expression of "positive definite" rational functions as sums of squares. The original question may be reformulated as:

Given a real multivariate polynomial that takes only non-negative values over the reals, can it be represented as a sum of squares of rational functions?

At the same time it is desirable, for certain questions as to the possibility of certain geometrical constructions, to know whether the coefficients of the forms to be used in the expression may always be taken from the realm of rationality given by the coefficients of the form represented.

Hilbert, 1888-1900

In 1888, Hilbert has shown that every definite form in n variables and degree $2d$ can be represented as sum of squares of other forms if and only if $\boxed{n = 2}$, or $\boxed{2d = 2}$ or $\boxed{n = 3,\ 2d = 4}$.

Hilbert's proof did not exhibit an explicit example: only in 1967 the first explicit example was constructed by Motzkin.

The Motzkin polynomial

$$f(x, y) = 1 + x^4y^2 + x^2y^4 - 3x^2y^2 = 1 + x^2y^2(x^2 + y^2 - 3)$$

cannot be represented as a sum of squares of other polynomials.

Its homogeneized form is

$$F(x, y, z) = z^6 + x^4y^2 + x^2y^4 - 3x^2y^2z^2$$

(here $n = 3$, $2d = 6$)

Artin-Schreier

Artin, Schreier. *Algebraische Konstruktion reeller Körper*. Abh. Math. Sem. Univ. Hamburg, 5(1): 85–99, (1927).

They invent the algebraic structure of **real closed fields** in order to describe in an axiomatic abstract way what are the algebraic properties of the real number field.

They give a positive answer to Hilbert (first part of the problem).

Tarski, Cohen, Hörmander

Tarski. *A decision method for elementary algebra and geometry*. (1951) (announced in 1931). University of California Press, Berkeley and Los Angeles, Calif.

Cohen. *Decision procedures for real and p-adic fields.* Comm. in Pure and Applied Math. 22, 131–151 (1969)

Hörmander. *The analysis of linear partial differential operators*. Berlin, Heidelberg, New-York, Springer (1983). 364–367.

Kreisel, Daykin, Delzell

Kreisel. *Sums of squares*. Summaries Summer Inst. Symbolic logic. Cornel Univ., 313–320. (1960).
http://hlombardi.free.fr/KREISEL-SOS.pdf

Daykin. *Hilbert's 17th problem*. Ph.D. Thesis, Univ. of Reading, (1961) unpublished, http://hlombardi.free.fr/Daykin-PhD-1961.pdf cited by **Kreisel**, *A survey of proof theory*. J. Symb. Logic 33, 321–388 (1968)

Delzell. *Kreisel's unwinding of Artin's proof*. 113–246, in Kreiseliana, (1996).

Krivine, Stengle

Krivine. *Anneaux préordonnés.* Journal d'analyse mathématique 12, 307–326 (1964).

Stengle. *A Nullstellensatz and a Positivstellensatz in semialgebraic Geometry*. Math. Ann. 207, 87–97, (1974).

Generalization and improvement of the solution of Hilbert's seventeenth problem.

A constructive solution for a discrete ordered field

Lombardi. *Une borne sur les degrés pour le Théorème des zéros réels effectif*. 323–345. In: Real Algebraic Geometry. Proceedings, Rennes (1991). http://hlombardi.free.fr/publis/ThZerosRennes.pdf

Coste, Lombardi, Roy. *Dynamical method in algebra: Effective Nullstellensätze*. Annals of Pure and Applied Logic, 111, 203–256. (2001) https://arxiv.org/abs/1701.05794

Lombardi. *Relecture constructive de la théorie d'Artin-Schreier*. Annals of Pure and Applied Logic. 91, (1998), 59–92. http://hlombardi.free.fr/publis/Rctas.pdf

The real number case

Delzell. *A continuous, constructive solution to Hilbert's 17th problem*. Inventiones Mathematicae, 76, 365–384. (1984).

González-Vega, Lombardi. *Nullstellensatz and Positivstellensatz for the Semipolynomials over an Ordered Field.* Journal of Pure and Applied Algebra. 90, 167–188. (1993).
http://hlombardi.free.fr/publis/PstSemiPols.pdf

Delzell, González-Vega, Lombardi. *A continuous and rational solution to Hilbert's 17th problem and several cases of the Positivstellensatz*. 61–75 in Progress in math No 109. (1993).
http://hlombardi.free.fr/publis/DGLMega92.pdf

Better complexity bounds

Lombardi, Perrucci, Roy. *An elementary recursive bound for effective Positivstellensatz and Hilbert 17-th problem*. (2015)
http://arxiv.org/abs/1404.2338

We prove an elementary recursive bound on the degrees for Hilbert 17-th problem, which is the expression of a nonnegative polynomial as a sum of squares of rational functions. More precisely, we obtain the following tower of five exponentials

$$2^{2^{2^{d^{4^{k}}}}}$$

where d is the degree and k is the number of variables of the input polynomial.

Related references

González-Vega, Lombardi. *Smooth parametrizations for several cases of the Positivstellensatz.* Math. Zeitschrift, 225, (1997), 427–451. http://hlombardi.free.fr/publis/SmoothPositivstellensatz.pdf

Lombardi. *Constructions cachées en algèbre abstraite (5) Principe local-global de Pfister et variantes.* International Journal of Commutative Rings. 2, (2003), 157–176.
http://hlombardi.free.fr/publis/LocalGlobalPfister.pdf

Coquand, Lombardi. *A logical approach to abstract algebra, a survey.* Math. Struct. in Comput. Science 16, (2006), 885–900.
http://hlombardi.free.fr/publis/AlgebraLogicCoqLom.pdf

Lombardi, Mahboubi. *Théories géométriques pour l'algèbre des nombres réels.* To appear.
https://hal.inria.fr/hal-01426164