Continuity properties for flat families of polynomials (I) Continuous parametrizations

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Abstract

Inspired by classical results in algebraic geometry, we study the continuity with respect to the coefficients, of the zero set of a system of complex homogeneous polynomials with a given pattern and when the Hilbert polynomial of the generated ideal is fixed. In this work we prove topological properties of some classifying spaces, e.g. the space of systems with given pattern, fixed Hilbert polynomial is locally compact, and we establish continuous parametrizations of Nullstellensatz formulae. In the general case we get local rational results but in the complex case we get global results using rational polynomials in the real and imaginary parts of the coefficients. In a second companion paper, we shall treat the continuity of zero sets for the Hausdorff distance, i.e., from a metric point of view.

Introduction

The long term purpose initiated by this paper is the rigorous construction of robust algorithms for approximate polynomial computations. In this direction a crucial task is to generalize to the multivariate case the continuity of the set of roots of a univariate polynomial with respect to its coefficients: Ostrowski's results in [26] give sharp bounds for the modulus of continuity. Indeed, the problem "compute the solutions of a system of equations" is ill–posed if the set of solutions does not depend continuously on the parameters (that are assumed to vary inside a well–described set) describing the system. In fact, continuity results are only a first step since in a computational setting we need a precise *modulus of continuity* in order to get a completely explicit computation.

In order to speak clearly about these continuity results we have to settle down more precisely our context. We choose here to work in the projective space and in the complex setting in order to avoid such phenomena as "roots that disappear at the infinity" or "real roots that disappear when becoming complex roots".

Let $K = \mathbb{P}^n(\mathbb{C})$, M a product of projective varieties and S a subset of M. K and M are compact metric spaces and $S \subset M$ is viewed as a "parameter space" presented by algebraic conditions. Let $(f_s)_{s\in S}$ be a family of homogeneous polynomial maps $f_s : \mathbb{C}^{n+1} \to \mathbb{C}^k$. We consider each f_s as a system of k equations with fixed degrees. To every $s \in S$, we associate the projective zero set $Z_s = \{x \in K : f_s(x) = 0\}$, which is compact.

The continuity of the map $s \mapsto Z_s$ with respect to the usual topology of compact subsets of K is particularly meaningful when the parameter space S is locally closed. So, we will establish this property of the parameter space in various (classical) geometric settings (this is done in section 2).

Then we will capture the idea of continuity of this map from several viewpoints specific to our algebraic context. In this direction, a first aim is to prove that the necessary condition: " the set

$$\{(g, s) : \, g \,\text{vanishes at the zeroes of}\, f_s\}$$

is closed in $G \times S$ " is true for simple spaces of functions G (see section 3.1).

When this is true for polynomial functions, a natural task is to establish a (local or global) continuous rational parametrization of the corresponding Nullstellensatz. This is done in section 3.2.

A second natural task is to prove the continuity of $s \mapsto Z_s$ with respect to the Hausdorff metric topology. This is the content of the second part of our work and will be presented in the companion paper [13].

Obviously these continuity results are not true without extra conditions. We aim at considering conditions on discrete algebraically computable invariants. We will say that a family $(f_s)_{s\in S}$ is flat if and only if the Hilbert polynomial of the spanned homogeneous ideals I_s is locally constant. Indeed, in commutative algebra and algebraic geometry (see [1], [18], [22]), the general algebraic notion of flatness has been designed in order to provide regular behavior, and in our setting the locally constancy of the Hilbert polynomial can be interpreted as the flatness of some natural morphism. We will prove that these flat families admit the desired continuity results.

As there are only a finite number of possible Hilbert polynomials for a given pattern (i.e. number of variables and list of degrees) of systems, they induce a finite stratification of S. This stratification refines the stratification of S by the continuity of Z_s that we aim to find.

To illustrate our choice of fixing the Hilbert polynomial, we consider the simple case of a pair of homogeneous bivariate polynomials. Let d_1 and d_2 be positive integers, $M = \mathbb{P}^{d_1}(\mathbb{C}) \times \mathbb{P}^{d_2}(\mathbb{C})$, S a subset of M and $K = \mathbb{P}^1(\mathbb{C})$. For any $s \in S$, f_s equals to two homogeneous polynomials in two variables with degrees d_1 and d_2, denoted by g_s and h_s. The constancy of the Hilbert polynomial of f_s means that the degree of the greatest common divisor of g_s and h_s is fixed. So, let S_k denotes the subset of M defined by:

$$s \in S_k \quad \iff \quad \deg(\gcd(g_s, h_s)) = k,$$

Then the gcd is obtained continuously from the coefficients in s as a (locally) fixed subresultant of the Sylvester matrix and by a classical argument (see [26], for example) we get the continuity of the map $s \in S_k \mapsto Z_s$. We note that the set S_k is locally closed and described by the vanishing of some subdeterminants of the Sylvester matrix of g_s and h_s and the non–vanishing of another subdeterminant, choosen among a fixed finite family.

We will see that in the general case the constancy of the Hilbert polynomial of the ideal generated by any polynomial system having the considered pattern will be characterized by the vanishing/non-vanishing of some subdeterminants extracted from a generalized Sylvester matrix. So, our technique of proof is rather elementary and effective.

The paper is organized as follows. Section 1 recalls several definitions and facts concerning Sylvester mappings and Hilbert polynomials. It also recalls some general bounds associated to algorithmic results in algebraic geometry, such as the Nullstellensatz or the solution of the membership problem. It also recalls some basic results in linear algebra and in real linear algebra. We insist on the fact that the real setting allows global parametrizations of the solutions when the rank of a linear system is known.

Section 2 gathers some definitions and useful facts about topological and semi-algebraic properties of classifying spaces for algebraic varieties. These results are derived from the existence of universal bounds given in section 1.

In section 3 we first provide elementary forms of continuity for belonging to the saturation of an homogeneous ideal when the Hilbert polynomial is known, and for vanishing on its zero set. Then we establish continuity results for parametrizations of corresponding membership problem and Nullstellensatz. In the complex case we get global parametrizations by rational polynomials in the real part and the imaginary part of the coefficients.

We notice that global continuous parametrizations of some instances of the complex Nullstellensatz provide a fully constructive version of (these instances of) the complex Nullstellensatz when dealing with polynomials given by Cauchy–complex numbers (i.e., given through rational approximations).

This introduction is finished by presenting some general considerations about *uniform bounds* in algebraic geometry. Many results in algebraic geometry have a constructive content and can be obtained through *uniform rational algorithms*. By a uniform rational algorithm we mean precisely:

1. The input of the algorithm is given by a finite list $\gamma = (\gamma_j)_{j \in J}$ of elements of a base field $\mathbf{K}$ (e.g., the coefficients of a finite list of polynomials having a given pattern).

2. The algorithm makes some rational computations, i.e., it computes some polynomials P_i in $\mathbb{Z}[(c_j)_{j \in J}]$. The only tests are tests " $P_i(\gamma) = 0$?". This introduces some branches in the computation: with a concrete input $\gamma \in \mathbf{K}^J$ we follow some precise branch of a computation tree.

3. For any field $\mathbf{K}$ and for any input in $\mathbf{K}^J$ the computation is finished in a finite number of steps: the output is given by a finite list of polynomials in $\mathbb{Z}[(c_j)_{j \in J}]$ and eventually by a finite number of booleans coding the answer of some tests inside the computation.

E.g., "solving a linear system", "computing the dimension of an algebraic variety" or "putting the variety in Noether position" are obtained through uniform rational algorithms when the pattern of the polynomial system is given.

We claim that the existence of such an algorithm implies the existence of uniform bounds on its length when the characteristic is fixed. I.e., bounds that are neither dependent on the field $\mathbf{K}$ nor on the input γ.

In fact, consider the γ_j's as indeterminates and "run the algorithm". You construct a big computation tree. Since the computation is finite for any concrete input γ, the computation tree could contain an infinite branch only if the system of equations and inequations corresponding to the branch is impossible. So we can improve the algorithm by testing this impossibility (by quantifier elimination in the algebraic closure of $\mathbf{K}$, which is a rational computation inside $\mathbf{K}$) and by stopping the corresponding branches at the node where the impossibility appears. These nodes are never attained for a concrete input.

This improved computation tree is a binary tree which produces the correct result for any concrete γ in any field $\mathbf{K}$. Let us see that this tree has only finite branches. Let us assume by contradiction an infinite branch. There is along this branch an infinte set of algebraic conditions $P_j(\gamma) = 0$, or $P_j(\gamma) \neq 0$. We know that no concrete input γ (in any field which has the given characteristic) satisfies all these conditions. This means that this set of conditions has no model, which implies (by the compactness theorem of logic) that a finite subset is inconsistent. Since in our construction we cut the branch when the corresponding system of conditions becomes inconsistent, the infinite branch cannot exist. So our tree is finite by Koenig's lemma (see e.g. [24] p. 7). And this gives some uniform bound.

In fact we see that the parameter space $\mathbf{K}^J$ is covered by a *finite* disjoint union of $\mathbf{K}_0$-Zariski basic locally closed subsets S_i corresponding to the branches of the tree ($\mathbf{K}_0$ is the prime field contained in $\mathbf{K}$). For each S_i the computation is *the same* for any input in S_i.

In practice an explicit uniform bound can always be obtained (without using compacteness theorem and Koenig's lemma, which are not constructive) through a close inspection of the termination proof, if this proof is constructive. E.g., Seidenberg's proofs in [27] give uniform bounds. In general much better bounds can be obtained by less direct and more sophisticated arguments.

We will assume the reader familiar with basics in effective algebraic geometry as described e.g. in the textbook [3].

1 Some algebro-geometric preliminaries

In this paper $\mathbf{K}$ is a field contained in an algebraically closed field $\mathbf{C}$ and $\mathbf{K}_0$ is the prime field contained in $\mathbf{K}$.

1.1 Hilbert Polynomial and Sylvester mappings

Let $\mathcal{J}$ be an ideal in $\mathbf{K}[x_0, \ldots, x_n] = \mathbf{K}[\mathbf{x}]$. $\mathcal{J}$ is said homogeneous if the relation $F \in \mathcal{J}$ implies that all the homogeneous components of F are in $\mathcal{J}$. If $\mathcal{J}$ is an homogeneous ideal in $\mathbf{K}[\mathbf{x}]$ then the saturation of $\mathcal{J}$ is the homogeneous ideal in $\mathbf{K}[\mathbf{x}]$ defined by

$$\overline{\mathcal{J}} = \left\{ g \in \mathbf{K}[\mathbf{x}] : \exists k \in \mathbb{N} \quad (\mathbb{I}(x_0, \ldots, x_n))^k \, g \subseteq \mathcal{J} \right\}.$$

Let $\mathbf{K}[\mathbf{x}]_\ell$ be the $\mathbf{K}$–vector space of homogeneous polynomials of total degree ℓ (including 0). If $\mathcal{J}$ is a homogeneous ideal and $\mathcal{J}_\ell = \mathcal{J} \cap \mathbf{K}[\mathbf{x}]_\ell$ then the *Hilbert function of* $\mathcal{J}$ is defined by

$$\mathrm{Hf}_{\mathcal{J}}(\ell) = \dim_{\mathbf{K}} \mathbf{K}[\mathbf{x}]_\ell / \mathcal{J}_\ell.$$

For ℓ sufficiently large $\overline{\mathcal{J}}_\ell = \mathcal{J}_\ell$ and $\mathrm{Hf}_{\mathcal{J}}(\ell)$ is a polynomial function (see chapter 9 in [3], for example), the *Hilbert polynomial of* $\mathcal{J}$, $\mathrm{Hp}_{\mathcal{J}}(\ell)$. The smallest ℓ for which this happens is called the *regularity* and there exists an explicit function $\Delta_1(n,d)$ providing the regularity for $\mathcal{J}$ (see [16]). The degree m of the Hilbert polynomial is the dimension of the projective variety in $\mathbb{P}^n(\mathbf{C})$ defined by $\mathcal{J}$ (note that the degrees of $\mathrm{Hp}_{\mathcal{J}}(\ell)$ and $\mathrm{Hp}_{\sqrt{\mathcal{J}}}(\ell)$ are the same). The *Poincaré series of* $\mathcal{J}$

$$P(t) = \sum_\ell \mathrm{Hf}_{\mathcal{J}}(\ell) t^\ell,$$

which codes the Hilbert function of $\mathcal{J}$, is always equal to a rational function.

Let X be a projective $\mathbf{K}$–variety in $\mathbb{P}^n(\mathbf{C})$. The *Hilbert polynomial of* X is defined as $\mathrm{Hp}_{\mathbb{I}(X)}(\ell)$ where $\mathbb{I}(X)$ is the homogeneous ideal spanned by the homogeneous polynomials in $\mathbf{K}[\mathbf{x}]$ vanishing on X.

Let $lst = (n, d_1, \ldots, d_p)$ be a list of positive integers. Let $\mathbf{f} = (f_1, \ldots, f_p)$ be a list of homogeneous $\mathbf{K}$–polynomials in $n+1$ variables with degrees $(d_1, \ldots, d_p)$. The linear mapping

$$\begin{array}{rccc} \mathrm{Syl}_{\mathbf{f}}: & \mathbb{B} = \mathbf{K}[\mathbf{x}]^p & \longrightarrow & \mathbf{K}[\mathbf{x}] \\ & (u_1, \ldots, u_p) & \longmapsto & u_1 f_1 + \ldots + u_p f_p \end{array}$$

is called the *Sylvester mapping for the polynomial system* $\mathbf{f}$. Given an integer $\ell \geq \max(d_1, \ldots, d_p)$, the linear map

$$\begin{array}{rccc} \mathrm{Syl}_{\mathbf{f},\ell}: & \mathbf{K}[\mathbf{x}]_{\ell-d_1} \times \cdots \times \mathbf{K}[\mathbf{x}]_{\ell-d_p} & \longrightarrow & \mathbf{K}[\mathbf{x}]_\ell \\ & (u_1, \ldots, u_p) & \longmapsto & u_1 f_1 + \cdots + u_p f_p \end{array}$$

is also called the *Sylvester mapping in degree* ℓ *for the polynomial system* $\mathbf{f}$. In fact $\mathrm{Syl}_{\mathbf{f},\ell}$ is the restriction of $\mathrm{Syl}_{\mathbf{f}}$ to the subspace $\mathbf{K}[\mathbf{x}]_{\ell-d_1} \times \cdots \times \mathbf{K}[\mathbf{x}]_{\ell-d_p}$ of $\mathbb{B}$. After a suitable choice of monomial bases in the above $\mathbf{K}$-vector spaces (free $\mathbf{K}$-modules in the general case), we get the so called *Sylvester matrix in degree* ℓ for the polynomial system $\mathbf{f}$. We denote this matrix by $\mathrm{Sylv}_{\mathbf{f},\ell}$. The corank of $\mathrm{Sylv}_{\mathbf{f},\ell}$ is equal to $\mathrm{Hf}_{\mathcal{J}}(\ell)$ where $\mathcal{J} = \mathbb{I}(\mathbf{f})$ is the homogeneous ideal of $\mathbf{K}[\mathbf{x}]$ generated by $f_1, \ldots, f_p$. We also denote by $\mathrm{ESyl}_{\mathbf{f},\ell}$ the image of the linear map $\mathrm{Syl}_{\mathbf{f},\ell}$.

Remark 1.1 In the affine case, one can define similarly Hilbert function and Hilbert polynomial, Sylvester mappings $\mathrm{Syl}^{\mathrm{aff}}_{\mathbf{f},\ell}$ and Sylvester matrices $\mathrm{Sylv}^{\mathrm{aff}}_{\mathbf{f},\ell}$: replace $\mathbf{K}[x_0, \ldots, x_n]_\ell$ by

$$\mathbf{K}[x_1, \ldots, x_n]_{\leq \ell} = \{g \in \mathbf{K}[x_1, \ldots, x_n] : \deg(g) \leq \ell\}.$$

For more details see chapter 9 in [3] .

1.2 Complete intersections

A list of homogeneous polynomials $\mathbf{f} = (f_1, \ldots, f_p)$ in $\mathbf{K}[\mathbf{x}]$ is said to be a *regular sequence* if the variety $\mathbf{V}(\mathbb{I}(\mathbf{f}))$ is non–empty and every f_i is a non zero divisor in $\mathbf{K}[\mathbf{x}]/\mathbb{I}(f_1, \ldots, f_{i-1})$. Then the corresponding variety or scheme is said to be a complete intersection.

If d_i is the degree of each f_i then the *associated Koszul complex* is defined as

$$\mathcal{K}: \quad 0 \to \bigwedge^p \mathbb{B} \xrightarrow{\partial_p} \bigwedge^{p-1} \mathbb{B} \xrightarrow{\partial_{p-1}} \cdots \xrightarrow{\partial_2} \bigwedge^1 \mathbb{B} \xrightarrow{\partial_1} \mathbf{K}[\mathbf{x}] \to 0$$

where $\mathbb{B}$ is the free $\mathbf{K}[\mathbf{x}]$–module $\mathbf{K}[\mathbf{x}]^p$ with basis $\{e_1, \ldots, e_p\}$ and the differentials ∂_k are defined by

$$\partial_k(e_{j_1} \wedge \cdots \wedge e_{j_k}) = \sum_{i=1}^k (-1)^{i+1} f_{j_i} e_{j_1} \wedge \cdots \wedge \widehat{e_{j_i}} \wedge \cdots \wedge e_{j_k}.$$

It is important to note here that the differential ∂_1 is $\mathrm{Syl}_{\mathbf{f}}$, the Sylvester mapping for the polynomial system $\mathbf{f}$. For the following theorem characterizing regular sequences in terms of the almost exactness of Koszul complex, see for example [8] or [28].

Theorem 1.2 *Let $\mathbf{f} = (f_1, \ldots, f_p)$ be a list of homogeneous polynomials in $\mathbf{K}[\mathbf{x}]$. Then the following assertions are equivalent:*

- $\mathbf{f}$ *is a regular sequence.*
- *the codimension of the zero set equals the length of the sequence.*
- *up to the surjectivity of ∂_1, the Koszul complex is exact in all degrees.*
- *the Poincaré series of $\mathbb{I}(\mathbf{f})$ verifies the equality:*

$$P(t) = \sum_{\ell} \mathrm{Hf}_{\mathbb{I}(\mathbf{f})}(\ell) t^{\ell} = \prod_i \frac{(1 - t^{d_i})}{(1 - t)^n}.$$

Note that in this case the Hilbert polynomial depends only on the degrees d_i of the generators f_i. Remark that this theorem can be seen as a generalization of Bezout's Theorem (which corresponds to the zero dimensional case).

1.3 Example

Along the paper we will illustrate the steps of our study with the following geometric example, which is simple but not trivial. We will denote this example by (E).

We take $K = \mathbb{P}^3(\mathbb{C})$, $M = \mathbb{P}^2(\mathbb{C})$ and $S = \mathbb{C}^2$, $s = (a, b)$, $\mathbf{f} = (f_1, f_2, f_3)$ with $f_1 = l_1 l_2$, $f_2 = l_1 c_2 + l_2 c_1 + a c_3$ and $f_3 = c_1 c_2$ where $l_1 = x$, $l_2 = x + bt$, $c_1 = x^2 + y^2 + z^2 - t^2$, $c_2 = x^2 + y^2 - z^2 - t^2$ and $c_3 = x^3 + xy^2 + z^3$. Geometrically, when $a = 0$ the zero set consists of the union of two conics $\Gamma_1 : (c_1 = l_1 = 0)$ and $\Gamma_2 : (c_2 = l_2 = 0)$. When $b = 0$ the two conics become coplanar. However when $a \neq 0$ the zero set consists of 24 points (when counted with multiplicities). It contains in particular the 12 points given by the intersection of the two conics with the cubic surface $c_3 = x^3 + xy^2 + z^3 = 0$. There are 6 more points on the conic $\Gamma_3 : (c_1 = l_2 = 0)$, and 6 last points on the conic $\Gamma_4 : (c_2 = l_1 = 0)$. These 12 last points depend on a.

We will use this example all along the text. The computations have been made with the general purpose computer algebra system `Maple` and the specialized one for Algebraic Geometry `Singular`.

We get the following results about the Hilbert polynomial of $\mathbf{f}$. For all the possible values of a and b the ideal generated by f_1, f_2 and f_3 is saturated. When $a \neq 0$ the Hilbert polynomial is constant, equal to 24, the Poincaré series is

$$\frac{t^6 + 3\,t^5 + 5\,t^4 + 6\,t^3 + 5\,t^2 + 3\,t + 1}{1 - t} = 1 + 4\,t + 9\,t^2 + 15\,t^3 + 20\,t^4 + 23\,t^5 + \sum_{\ell=6}^{\infty} 24\,t^{\ell}$$

and the regularity equals 6. It is a complete intersection. When $a = 0$, $b \neq 0$ the Hilbert polynomial is equal to $4\ell + 2$, the Poincaré series is

$$\frac{t^7 - 2\,t^5 - t^4 + t^3 + 2\,t^2 + 2\,t + 1}{(1 - t)^2} = -1 - 2\,t - t^2 + t^3 + 2\,t^4 + t^5 + \frac{2t + 2}{(1 - t)^2}$$

(where $\frac{2t+2}{(1-t)^2} = \sum_{\ell=0}^{\infty} (4\ell + 2) t^{\ell}$) and the regularity equals 6. When $a = b = 0$ the Hilbert polynomial is equal to $4\ell + 6$, the Poincaré series is

$$-\frac{t^6 + t^5 - t^3 - 2\,t^2 - 2\,t - 1}{(1 - t)^2} = -5 - 6\,t - 5\,t^2 - 3\,t^3 - t^4 + \frac{-2t + 6}{(1 - t)^2}$$

(where $\frac{-2t+6}{(1-t)^2} = \sum_{\ell=0}^{\infty} (4\ell + 6) t^{\ell}$) and the regularity equals 5.

The real affine part $(t = 1)$ for the particular case $a = 1, b = 1$ appears in Figure 1. All the solutions of $\mathbf{f}$ in this case are affine and only 14 out of the 24 complex solutions are real (taking into account multiplicities): the considered system has only 6 different real solutions: 4 in the plane $x = 0$ (two of them with multiplicity 5 each) and 2 in the plane $x = -1$.

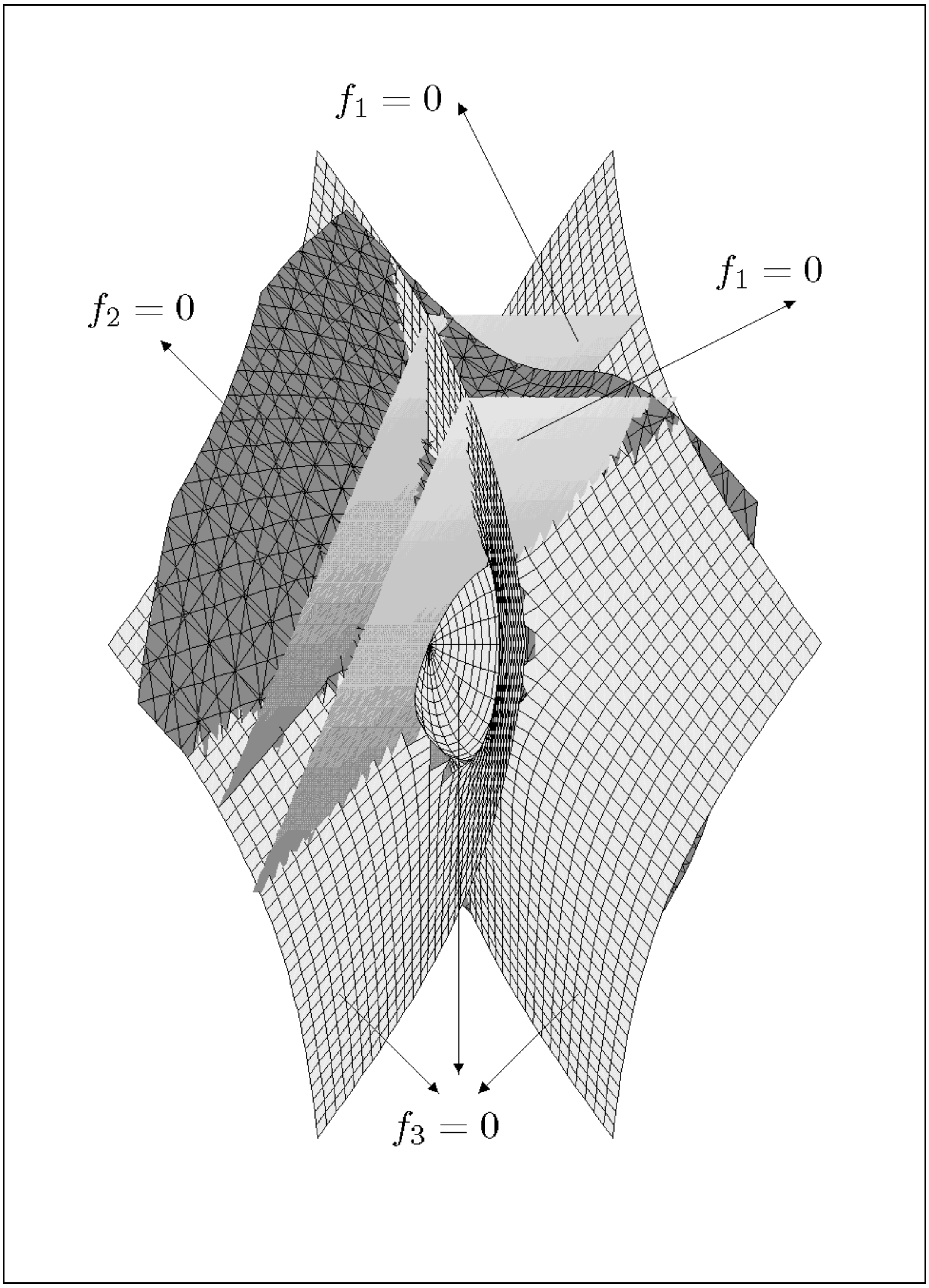

Figure 1: The real affine ($t = 1$) part of the case $a = 1$ and $b = 1$.

1.4 Uniform bounds related to the Nullstellensatz

In this section we present bounds that will be used in order to chose the suitable degree ℓ for the Sylvester matrix $\mathrm{Sylv}_{\mathbf{f},\ell}$ in sections 2 and 3: uniform bounds relative to the membership problem for a polynomial ideal (or its radical or its saturation) and to the effective Nullstellensatz are summarized (see [27], [2], [23], [21] and the references contained therein, [17]).

Let $\mathbb{I}(\mathbf{f})$ be the ideal generated by $f_1, \ldots, f_p$ in $\mathbf{K}[x_1, \ldots, x_n]$ such that d is an upper bound of the f_i's total degrees. Then:

1. There exists an explicit function $\Delta_2(n, d)$ verifying that the polynomial h belongs to $\mathbb{I}(\mathbf{f})$ if and only if there exist polynomials $g_1, \ldots, g_p$ in $\mathbf{K}[\mathbf{x}]$ such that

$$h = \sum_{j=1}^{p} g_j f_j$$

with $\max\{\deg(g_j f_j) : j \in \{1, \ldots, p\}\} \leq \Delta_2(n, d) + \deg(h)$.

2. There exist two explicit functions $\Delta_2(n, d)$ and $N_2(n, d)$ verifying that the polynomial h belongs to $\sqrt{\mathbb{I}(\mathbf{f})}$ if and only if there exist polynomials $g_1, \ldots, g_p$ in $\mathbf{K}[\mathbf{x}]$ such that

$$h^N = \sum_{j=1}^{p} g_j f_j$$

with $N \leq N_2(n, d)$ and $\max\{\deg(g_j f_j) : j \in \{1, \ldots, p\}\} \leq \Delta_2(n, d) + N \deg(h)$.

Let $\mathbb{I}(\mathbf{f})$ be an homogeneous ideal generated by $f_1, \ldots, f_p$ in $\mathbf{K}[x_0, \ldots, x_n]$ such that d is an upper bound of the f_i's total degrees. Then:

1. There exists an explicit function $N_1(n, d)$ verifying that the homogeneous polynomial h belongs to $\overline{\mathcal{J}}$ if and only if there exist homogeneous polynomials $g_{i1}, \ldots, g_{ip}$ in $\mathbf{K}[\mathbf{x}]$ such that for each $i = 0, \ldots, n$ we get an homogeneous equality

$$h\, x_i^N \;=\; \sum_{j=1}^{p} g_{ij} f_j$$

with $N = N_1(n, d)$.

2. There exists an explicit function $N_3(n, d)$ verifying that the homogeneous polynomial h belongs to $\sqrt{\mathbb{I}(\mathbf{f})}$ if and only if there exist homogeneous polynomials $g_1, \ldots, g_p$ in $\mathbf{K}[\mathbf{x}]$ with an homogeneous equality

$$h^N = \sum_{j=1}^{p} g_j f_j$$

with $N \leq N_3(n, d)$.

Let us note that the bounds related to the regularity of the Hilbert polynomial and the membership problem are double exponential while those related to the Nullstellensatze are simple exponential.

Finally, let us indicate that "in the other direction", one can attach to each Hilbert polynomial H a pattern lst such that if $\mathbb{I}(\mathbf{f})$ has H as Hilbert polynomial, then there exists a system $\mathbf{g}$ with pattern lst such that the quotient graded algebra corresponding to $\mathbb{I}(\mathbf{f})$ and $\mathbb{I}(\mathbf{g})$ are isomorphic (this implies that they have the same Hilbert polynomial). This property is used to define an algebraic structure on the so-called Hilbert Scheme. See e.g. [30], [19], [20] for an introductory discussion on that point.

1.5 Some general linear algebra

Let $\mathbf{K}_0$ be a prime field contained in a field $\mathbf{K}$. We give some lemmas translating simple facts of linear algebra to geometric conditions. These results will be used for the analysis of Sylvester matrices depending on parameters.

Fixing ranks defines Zariski locally closed sets
Next it is shown how to describe, in the coefficient space, several conditions on the rank of a matrix plus the exactness of a sequence of linear mappings. The corresponding sets are then analyzed with respect to the Zariski topology in the considered coefficient space.

Lemma 1.3 (for a matrix, being of rank s is a locally closed condition)
Let $\mathbf{K}^{q\times r}$ be the coefficient space of $(q \times r)$–matrices. Let s be an integer $\leq \min(q,r)$. Then
— the matrices of rank $\leq s$ form a $\mathbf{K}_0$–Zariski closed subset of $\mathbf{K}^{q\times r}$,
— the matrices of maximal rank $\min(q,r)$ form a $\mathbf{K}_0$–Zariski open subset of $\mathbf{K}^{q\times r}$,
— the matrices of constant rank s form a $\mathbf{K}_0$–Zariski locally closed subset of $\mathbf{K}^{q\times r}$.

Proof.
The first claim follows from the characterization: all $(s+1)$–minors are zero. The other claims follow immediately. □

Notation 1.4 Given a field $\mathbf{K}$ and three positive integers q, r, s we denote $\mathcal{M}_{q,r,s}(\mathbf{K})$ the subset of $\mathbf{K}^{q\times r}$ whose elements are the matrices of rank s.

Lemma 1.5 (for a complex, being exact is an open condition)
Consider general matrices for linear maps $\delta_1, \ldots, \delta_q$ ($\delta_i : \mathbf{K}^{r_i} \rightarrow \mathbf{K}^{r_{i+1}}$). The condition " $\delta_{i+1} \circ \delta_i = 0$ for $i = 1, \ldots, p$ " defines a $\mathbf{K}_0$–Zariski closed subset of the coefficient space $\mathbf{K}^{r_2\times r_1} \times \cdots \times \mathbf{K}^{r_{q+1}\times r_q}$. Inside the previous variety, the exactness of the sequence

$$0 \rightarrow \mathbf{K}^{r_1} \xrightarrow{\delta_1} \mathbf{K}^{r_2} \xrightarrow{\delta_2} \cdots \xrightarrow{\delta_q} \mathbf{K}^{r_{q+1}} \quad (*)$$

defines a $\mathbf{K}_0$–Zariski open subset.

Proof.
The conditions "$\delta_{i+1} \circ \delta_i = 0$" clearly define an algebraic subvariety of the considered coefficient space. Let us prove that in this variety the exactness of the sequence $(*)$ is an open condition. The condition "$\mathrm{Ker}(\delta_1) = 0$" is Zariski open (some minor of order r_1 is nonzero). Under this condition we have $\dim(\mathrm{Im}(\delta_1)) = r_1$. The condition " $\mathrm{Ker}(\delta_2) = \mathrm{Im}(\delta_1)$ " is then equivalent to $\mathrm{rank}(\delta_2) \geq r_2 - r_1$ and this is an open condition. Under this condition we have $\dim(\mathrm{Im}(\delta_2)) = r_2 - r_1$. The condition "$\mathrm{Ker}(\delta_3) = \mathrm{Im}(\delta_2)$" is then equivalent to $\mathrm{rank}(\delta_3) \geq r_3 - r_2 + r_1$ and this is an open condition. And so on. □

Continuity and local rational parametrization for linear algebra with fixed ranks
Next lemma shows how to parametrize the membership of a vector to the image of a linear mapping.

Lemma 1.6 (Cramer conditions and Cramer formulae)
Let q, r, s be positive integers with $s \leq \min(q,r)$. Let $\mathbf{K}^{q\times r} \times \mathbf{K}^q$ be the coefficient space of couples (M, g) where M is a $(q \times r)$–matrix and g a column vector. Then

1. *the condition " $g \in \mathrm{Im}(M)$ " is closed over $\mathcal{M}_{q,r,s}(\mathbf{K})$ (the subset of rank s matrices), and*
2. *there is a finite $\mathbf{K}_0$–Zariski open cover $(U_i)_{1\leq i\leq m}$ (with $m = \binom{q}{s} \times \binom{r}{s}$) of $\mathcal{M}_{q,r,s}(\mathbf{K})$ s.t. over each U_i there is a $\mathbf{K}_0$–polynomial description for $g \in \mathrm{Im}(M)$, i.e., there are $\mathbf{K}_0$–polynomials $t_i(M)$ and $t_{ij}(M, g)$ ($1 \leq i \leq m$, $1 \leq j \leq r$) s.t. for all $M \in \mathcal{M}_{q,r,s}$ and $g \in \mathrm{Im}(M)$ we have*
   - *$t_i(M)$ is nowhere zero on U_i and*
   - *$t_i(M)\, g = M\, (t_{i1}(M, g), \ldots, t_{ir}(M, g))^T$.*

Proof.
Linear algebra routines and Cramer formulae give the desired results. □

The meaning of the last claim is that a solution of a linear system can be given locally by uniform continuous rational expressions when the rank of the system is fixed (and some solution does exist).

Now we give an algebraic continuity lemma for quotient structures. We establish a lemma for a structure given by a bilinear map between finite dimensional vector spaces. Same lemma, with similar proof, is valid for linear or multilinear maps.

Lemma 1.7 (Continuity for "viewing a bilinear map in quotient spaces of fixed dimension")
Let $lst = (q, q', q'', r, r', r'', s, s', s'')$ be a list of positive integers with $s \le q, r$, $s' \le q', r'$ and $s'' \le q'', r''$. Let $E = \mathbf{K}^q$, $E' = \mathbf{K}^{q'}$ and $E'' = \mathbf{K}^{q''}$ be three finite dimensional vector spaces. Let $\mathcal{B}_{q,q',q''}(\mathbf{K}) = \mathbf{K}^{q\times q'\times q''}$ the coefficient space for $\mathbf{K}$–bilinear maps $\varphi : E \times E' \to E''$ and consider matrices $M \in \mathcal{M}_{q,r,s}(\mathbf{K})$, $M' \in \mathcal{M}_{q',r',s'}(\mathbf{K})$, $M'' \in \mathcal{M}_{q'',r'',s''}(\mathbf{K})$, $F \in \mathcal{B}_{q,q',q''}(\mathbf{K})$ verifying

$$F(E \times \mathrm{Im}(M'))) \subset \mathrm{Im}(M'') \quad \text{and} \quad F(\mathrm{Im}(M) \times E')) \subset \mathrm{Im}(M'').$$

Then:

- *Such quadruples of matrices (M, M', M'', F) form a $\mathbf{K}_0$–Zariski locally closed set $S_{lst}(\mathbf{K})$ (it is a subset of $\mathbf{K}^t$ with $t = qr + q'r' + q''r'' + qq'q''$).*
- *For $(M, M', M'', F) \in S_{lst}(\mathbf{K})$ consider the bilinear map obtained from F in quotient spaces*

$$F_{M,M',M''} : (E/\mathrm{Im}(M)) \times (E'/\mathrm{Im}(M')) \longrightarrow E''/\mathrm{Im}(M''),$$

  then there exists a finite Zariski open covering (V_i) of $S_{lst}(\mathbf{K})$ such that on each V_i, we can give explicitly "fixed" bases of quotient spaces $E/\mathrm{Im}(M)$, $E'/\mathrm{Im}(M')$, $E''/\mathrm{Im}(M'')$. Moreover the corresponding constants of structure are given by fixed rational functions in the parameters, whose denominators are nowhere vanishing on V_i.

Proof.
An open set V_i is defined by the fact that the matrices M, M', M'' have a given coordinate subspace of $\mathbf{K}^q, \mathbf{K}^{q'}, \mathbf{K}^{q''}$ as supplementary subspace of their images.
Next, for any given such open subset, the structure of the corresponding $F_{M,M',M''}$ can be made explicit on the bases of the supplementary subspaces we have chosen. Each constant of structure (i.e., each entry of the matrix expressing $F_{M,M',M''}$ on the chosen bases of quotient spaces) is the unique solution of some invertible linear system written from F, M'' and the chosen coordinate bases of supplementary subspaces. Therefore by lemma 1.6, each constant of structure is given by suitable rational functions. □

Having no zero divisor is an open condition
Let us assume $\mathbf{K} \subset \mathbf{C}$ with $\mathbf{C}$ algebraically closed. We give two lemmas expressing the idea that, for a varying $\mathbf{C}$–algebra having some fixed form, being a domain is an open condition.

A $\mathbf{K}$–constructible set in $\mathbf{C}^m$ is defined by a finite Boolean combination on the atomic formulae $h \neq 0$ or $g = 0$ where h and g are $\mathbf{K}$–polynomials. Quantifier elimination over algebraically closed fields assures that the truth set in $\mathbf{C}^m$ for any first-order $\mathbf{K}$–formula with m free variables is a $\mathbf{K}$–constructible set in $\mathbf{C}^m$.

In the following, we use the well-known following fact (see e.g. [25] for the open case):

Proposition 1.8 *Let $\mathbf{K}$ be an ordered subfield of a real closed field $\mathbf{R}$, $\mathbf{L} = \mathbf{K}[\sqrt{-1}]$ and $\mathbf{C} = \mathbf{R}[\sqrt{-1}]$. If U is a $\mathbf{L}$–constructible and Euclidean open (resp. locally closed) set in $\mathbf{C}^n$ then U is also a $\mathbf{L}$–Zariski open (resp. locally closed) set.*

The two following lemmas are given for the zero characteristic case.

Lemma 1.9 *Assume that the characteristic of $\mathbf{K}$ is zero. Let $E = \mathbf{C}^q$, $F = \mathbf{C}^r$ and $G = \mathbf{C}^s$ be three finite dimensional vector spaces and $\mathcal{B}_{q,r,s}(\mathbf{C}) = \mathbf{C}^{q\times r\times s}$ the coefficient space for $\mathbf{C}$–bilinear maps $\varphi : E \times F \to G$. Then the subset of bilinear maps "without zero divisor" (i.e., $x \in \mathbf{C}^q$, $y \in \mathbf{C}^r$, $x \neq 0$ and $y \neq 0$ implies $\varphi(x, y) \neq 0$) is $\mathbf{K}_0$–Zariski open.*

Proof.
We have $\mathbf{K}_0 = \mathbb{Q}$ and we may assume that $\mathbf{C} = \mathbf{R}[\sqrt{-1}]$ with $\mathbf{R}$ real closed. Consider the three vector spaces E, F and G endowed with hermitian norms. Having no zero divisor means that

$$|\varphi(x,y)| > 0 \quad \text{for all } (x,y) \in E \times F \text{ s.t. } |x| = |y| = 1.$$

Since the spheres are compact in the semialgebraic meaning, for any such φ this implies

$$\exists c \in \mathbf{R}^{>0} \;\; \forall x \in E \, \forall y \in F \quad (\; (|x| = |y| = 1) \;\; \Rightarrow \;\; |\varphi(x,y)| \; \geq \; c \;),$$

and for each ψ sufficiently close to φ this implies

$$\forall x \in E \, \forall y \in F \quad (\; (|x| = |y| = 1) \;\; \Rightarrow \;\; |\psi(x,y)| \; \geq \; c/2 \;).$$

So the considered set is open in the Euclidean topology. As it is a $\mathbb{Q}$–constructible set in $\mathbf{C}^t$, by using Proposition 1.8, it is also Zariski open. □

We give now a more sophisticated lemma, with the same intuitive meaning, in the case where some bilinear map acts on variable quotient spaces.

Lemma 1.10 *Assume that the characteristic of* $\mathbf{K}$ *is zero. Let* $q, q', q'', r, r', r'', s, s', s''$ *be positive integers with* $s \leq q, r$, $s' \leq q', r'$ *and* $s'' \leq q'', r''$*. Let* $E = \mathbf{C}^q$, $E' = \mathbf{C}^{q'}$ *and* $E'' = \mathbf{C}^{q''}$ *be three finite dimensional vector spaces and*

$$\psi \; : \; E \times E' \; \rightarrow \; E''$$

be a fixed $\mathbf{C}$*–bilinear map. Consider matrices* $M \in \mathcal{M}_{q,r,s}(\mathbf{C})$, $M' \in \mathcal{M}_{q',r',s'}(\mathbf{C})$, $M'' \in \mathcal{M}_{q'',r'',s''}(\mathbf{C})$ *verifying*

$$\psi(E \times \mathrm{Im}(M'))) \subset \mathrm{Im}(M'') \quad \text{and} \quad \psi(\mathrm{Im}(M) \times E)) \subset \mathrm{Im}(M'').$$

From lemma 1.7 such triples of matrices (M, M', M'') *form a* $\mathbf{K}_0$*–Zariski locally closed set* $S(\mathbf{C})$*. For* $(M, M', M'') \in S(\mathbf{C})$ *consider the bilinear map obtained from* ψ *in quotient spaces*

$$\psi_{M,M',M''} \; : \; (E/\mathrm{Im}(M)) \times (E'/\mathrm{Im}(M')) \;\; \rightarrow \;\; E''/\mathrm{Im}(M'').$$

Then the set of (M, M', M'') *such that the bilinear map* $\psi_{M,M',M''}$ *has no zero divisor is* $\mathbf{C}$*–Zariski open in* $S(\mathbf{C})$*.*

Proof.
On each Zariski open subset given in lemma 1.7 apply lemma 1.9. □

1.6 Some real linear algebra

In this section we investigate our problem in a general semi-algebraic setting. Continuity issues are related to the Euclidean topology.

Let $\mathbf{K}$ be an ordered subfield of a real closed field $\mathbf{R}$, $\mathbf{L} = \mathbf{K}[\sqrt{-1}]$ and $\mathbf{C} = \mathbf{R}[\sqrt{-1}]$. We begin with a simple but fundamental example.

Example 1.11 Linear forms. Consider the following linear problem in elementary projective geometry: if the point a belongs to a linear subspace containing points $b_1, \ldots, b_p$ then make explicit this fact.
In its dual form: if the linear form g vanishes at the common zeroes of the linear forms $f_1, \ldots, f_p$ then express g as a linear combination of $f_1, \ldots, f_p$.
This problem of numerical matrix analysis is well-known not having any reasonable solution unless the rank of the corresponding matrix is known. When the rank is known, the problem has a uniform solution given in lemma 1.15.

We give here a very useful definition for controlling real phenomena in a complex setting.

Definition 1.12 *We call a function* $\Phi : \mathbf{C}^m \to \mathbf{C}$, $\gamma = (\gamma_1, \ldots, \gamma_m) \mapsto \Phi(\gamma)$ *a* **K**–real polynomial *if it is a* **K**–*polynomial in the real and imaginary part of the* γ_i*'s. So it is defined by two polynomials* $f, g \in \mathbf{K}[a_1, \ldots, a_m, b_1, \ldots, b_m]$ *such that*

$$f(\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_m) + \sqrt{-1}g(\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_m) = \Phi(\gamma_1, \ldots, \gamma_m)$$

if $\gamma_i = \alpha_i + \sqrt{-1}\beta_i$ *for* $i = 1, \ldots, m$*. This ring of functions will be denoted by* $\mathbf{K}\lfloor c_1, \ldots, c_m \rceil$ *where the* c_i*'s are formal variables.*

Let us remark that **K**–real polynomials are usually not **C**–polynomials. E.g. the mapping $\gamma \mapsto \overline{\gamma}$ is a **K**–real polynomial but not a **C**–polynomial.

Using sums of squares of absolute values, a **K**–Zariski closed set in a coefficient space $\mathbf{C}^m$ can always be defined as the zero set of a single **K**–real polynomial. In linear algebra this takes the form of Gram coefficients associated to a matrix.

In the following we see always a complex space $\mathbf{C}^m$ as endowed with an hermitian norm. From a real point of view, this is also a real space $\mathbf{R}^{2m}$ with an Euclidean norm.

Definition 1.13 *If* M *is a matrix in* $\mathbf{C}^{q \times r}$ *(representing a linear map between hermitian spaces) we denote by* $M^\star$ *the transpose of the conjugate of* M*. The matrix* $MM^\star$ *is hermitian nonnegative. The* Gram coefficients of M *are* $\mathcal{G}_k(M) = a_k$ *given by the formula*

$$\det(\mathrm{I}_q \ + \ T\, MM^\star) \ = \ 1 \ + \ a_1\, T \ + \ \cdots \ + \ a_q\, T^q.$$

So, $\mathcal{G}_k(M) = a_k$ is the $\mathbb{Q}$–real polynomial in the entries of M equal to the sum of squares of absolute values of all minors of order k in M. We define also

$$\mathcal{G}_0(M) = 1 \quad \text{and} \quad \mathcal{G}_t(M) = 0 \quad \text{for} \quad t > q.$$

We have also

$$\mathrm{Im}(M) \ = \ \mathrm{Im}(MM^\star) \ \text{ is the orthogonal space of } \mathrm{Ker}(MM^\star).$$

If φ_1 denotes the restriction of $MM^\star$ to $\mathrm{Im}(M)$ and if $s = \dim(\mathrm{Im}(M))$ then $a_s = \mathcal{G}_s(M) \neq 0$ and $\det(\mathrm{Id}_{\mathrm{Im}(M)} + T\varphi_1) \ = \ 1 + a_1T + \cdots + a_sT^s$. So by Cayley–Hamilton Theorem we get $\varphi_1^s - a_1\varphi_1^{s-1} + \cdots + (-1)^s a_s \mathrm{Id}_{\mathrm{Im}(M)} \ = \ 0$, φ_1 is invertible and

$$\varphi_1^{-1} \ = \ a_s^{-1}(a_{s-1}\mathrm{Id}_{\mathrm{Im}(M)} - a_{s-2}\varphi_1 + \cdots + (-1)^{s-1}\varphi_1^{s-1})\ .$$

Lemma 1.3 has the following global version:

Lemma 1.14 (Gram conditions for the rank, orthogonal projection on the image)
Let $\mathbf{C}^{q \times r}$ *be the coefficient space of* $(q \times r)$*–matrices. Then*

1. *the matrices of constant rank* s *form a Zariski locally closed subset of* $\mathbf{C}^{q \times r}$ *defined by the real conditions* $\mathcal{G}_s(M) \neq 0$, $\mathcal{G}_{s+1}(M) = 0$,

2. *the matrix* π_M *of the orthogonal projection on* $\mathrm{Im}(M)$ *can be expressed in the form*

$$\pi_M = a_s^{-1}(a_{s-1}MM^\star - a_{s-2}(MM^\star)^2 + \cdots + (-1)^{s-1}(MM^\star)^s),$$

where $a_k = \mathcal{G}_k(M)$*.*

We give now a fundamental lemma of numerical matrix analysis, which assures a drastic improvement of lemma 1.6.

Lemma 1.15 (Moore-Penrose inverse)
Let q, r, s *be positive integers with* $s \leq \min(q, r)$*. Let* $\mathbf{K}^{q \times r} \times \mathbf{K}^q$ *be the coefficient space of couples* (M, g) *where* M *is a* $(q \times r)$*–matrix and* g *a column vector. Then*

1. *the condition "* $g \in \mathrm{Im}(M)$ *" is Zariski closed over* $\mathcal{M}_{q,r,s}(\mathbf{C})$ *(the subset of rank* s *matrices) and is defined by the real condition* $\mathcal{G}_{s+1}(M|g) = 0$ *where* $M|g$ *denotes the* $(q \times (r+1))$*-matrix obtained by adding to* M *the column vector* g*, and*

2. *if* $R_{q,r,s}(M) = (a_{s-1}\mathrm{I}_r - a_{s-2}M^\star M + \cdots + (-1)^{s-1}(M^\star M)^{s-1})\, M^\star$ *in* $\mathbf{C}^{r\times q}$*, with* $a_k = \mathcal{G}_k(M)$*, then its entries are* $\mathbb{Q}$*–real polynomials in the entries of* M*, and*

$$a_s\, g \;=\; M\, R_{q,r,s}(M)\, g$$

for all $M \in \mathcal{M}_{q,r,s}$ *and* $g \in \mathrm{Im}(M)$*.*

The meaning of the last claim is that a solution of a linear system $Mx = g$ (with x unknown) can be given globally by a uniform continuous rational expression $x = a_s^{-1}\, R_{q,r,s}(M)\, g$ when the rank of the matrix M is equal to s (and some solution does exist).

2 Topological properties of some classifying spaces

2.1 Notations concerning the coefficient spaces of homogeneous polynomial systems

We fix some notations concerning the coefficient spaces for systems of homogeneous polynomials for the sequel. Let $\mathbf{K}_0 \subset \mathbf{K} \subset \mathbf{C}$ be three fields where $\mathbf{K}_0$ is a prime field and $\mathbf{C}$ is algebraically closed.

Let $lst = (n, d_1, \dots, d_p)$ be a list of positive integers, θ_i be equal to $\theta(n, d_i) = \binom{n+d_i}{n}$ and

$$\Theta = \theta_1 + \cdots + \theta_p = \theta(lst).$$

We consider $f_i(\mathbf{c}_i, \mathbf{x})$ the general homogeneous polynomial in variables $(x_0, \dots, x_n)$ of $\mathbf{x}$–degree d_i with coefficients $\mathbf{c}_i = (c_1^{(i)}, \dots, c_{m_i}^{(i)})$. For $\gamma \in \mathbf{K}^\Theta$ let $f_i^{[\gamma]}(\mathbf{x}) = f_i(\gamma, \mathbf{x})$ and $\mathbf{f}^{[\gamma]} = (f_1^{[\gamma]}, \dots, f_p^{[\gamma]})$.

So, the space $\mathbf{K}^\Theta$ is the *coefficient space* for a general system $\mathbf{f}^{[\gamma]} = (f_1^{[\gamma]}, \dots, f_p^{[\gamma]})$ of p homogeneous polynomials in variables $(x_0, \dots, x_n)$ with $\mathbf{x}$–degrees $d_1, \dots, d_p$. We shall call the list lst the *pattern of a system of homogeneous polynomials* in this coefficient space $\mathbf{K}^\Theta$ ($\Theta = \theta(lst)$). We denote by $\mathbb{I}(\mathbf{f}^{[\gamma]})$ the homogeneous ideal in $\mathbf{K}[\mathbf{x}]$ generated by the polynomials $f_i(\gamma, \mathbf{x})$.

Let us assume w.l.o.g. that the f_i's are not all identically zero. Then we can consider γ as an element of the *projective coefficient space* $\mathbb{P}^{\Theta-1}(\mathbf{K})$. Let now H be a polynomial that appears as the Hilbert polynomial of some homogeneous ideal $\mathbb{I}(\mathbf{f}^{[\gamma]})$.

Notation 2.1
We denote by $\mathcal{HP}_{n,d_1,\dots,d_p,H}(\mathbf{K}) = \mathcal{HP}_{lst,H}(\mathbf{K})$ the subset of the projective coefficient space $\mathbb{P}^{\Theta-1}(\mathbf{K})$ made of the polynomial systems $\mathbf{f}^{[\gamma]}$ giving an ideal $\mathbb{I}(\mathbf{f}^{[\gamma]})$ with Hilbert polynomial H.
We denote also by $\Delta_{\mathrm{HP}}(lst)$ an integer D_0 s.t. the Sylvester matrix in each degree $d \geq D_0$ has corank $H(d)$ for any H and any $\mathbf{f}^{[\gamma]} \in \mathcal{HP}_{lst,H}(\mathbf{K})$ (it is enough to take D_0 bigger than the regularity of all these $\mathbb{I}(\mathbf{f}^{[\gamma]})$).

2.2 Having a given Hilbert polynomial is a locally closed condition

Theorem 2.2 (Having a given Hilbert polynomial is a locally closed condition)
Consider polynomial systems $\mathbf{f}^{[\gamma]}(\mathbf{x})$ *in* $\mathcal{HP}_{lst,H}(\mathbf{K})$*, i.e. with fixed pattern lst and Hilbert polynomial* H*. Then the subset* $\mathcal{HP}_{lst,H}(\mathbf{K})$ *of the coefficient space* $\mathbf{K}^\Theta$ *is* $\mathbf{K}_0$*–constructible and Zariski locally closed.*

Proof.
Having H as Hilbert polynomial is equivalent to: the Sylvester matrix of $\mathbf{f}^{[\gamma]}(\mathbf{x})$ has corank $H(d)$ in degrees $d = D_0, D_0+1, \dots, D_0+n$ for $D_0 = \Delta_{\mathrm{HP}}(n, d_1, \dots, d_p)$. So apply lemma 1.3. □

The following two results are of special interest.

Theorem 2.3 (emptiness of the associated variety is an open condition)
The condition

$$\mathbf{f}^{[\gamma]} \text{ defines the empty projective variety}$$

is an open condition. More precisely, the set

$$\{\gamma \ : \ \mathbf{f}^{[\gamma]} \text{ has no zero in } \mathbf{C}^n \setminus \{\mathbf{0}\}\} \ \subset \ \mathbf{K}^\Theta$$

is Zariski open in $\mathcal{HP}_{lst,H}$ *(nonempty iff* $p \geq n+1$*).*

Proof.
Having no projective zero is equivalent to: the Sylvester matrix of $\mathbf{f}^{[\gamma]}(\mathbf{x})$ has maximal rank in degree $D_0 = \Delta_{\mathrm{HP}}(lst)$. So apply lemma 1.3. □

Theorem 2.4 (being a regular sequence is an open condition)
The condition

$$\mathbf{f}^{[\gamma]} \text{ is a regular sequence}$$

is an open one. More precisely, the set

$$\{\gamma \ : \ \mathbf{f}^{[\gamma]} \text{ is a regular sequence } \} \ \subset \ \mathbf{K}^\Theta$$

is Zariski open in $\mathcal{HP}_{lst,H}$ *(nonempty iff* $p \leq n+1$*).*

Proof.
Being a regular sequence is equivalent to: the Koszul complex is exact in degrees $D_0, D_0+1, \ldots, D_0+p$ for $D_0 = \Delta_{\mathrm{HP}}(lst)$. So apply lemma 1.5. □

Remark 2.5 Theorem 2.3 corresponds to Hilbert polynomial equals to 0 and theorem 2.4 to Hilbert polynomial fixed and given by the Poincare series

$$\frac{\prod_i (1 - z^{d_i})}{(1-z)^n}$$

where d_i denotes the degree of the generator f_i.

Remark 2.6 For a given pattern lst, there exist only finitely many Hilbert polynomials possible. These polynomials can be ordered, according to their behavior at infinity. We get in this way an ordered list of Hilbert polynomials: $H_1, \ldots, H_r$ (depending on lst). For some sufficiently high degree Δ we have $H_1(\Delta) < \cdots < H_r(\Delta)$. Assume $1 \leq n_1 \leq n_2 \leq r$. Then the fact that the Hilbert polynomial of a system $\mathbf{f}^{[\gamma]}$ belongs to $[H_{n_1}, H_{n_2}]$ defines a locally closed set in $\mathbb{P}^{\Theta-1}(\mathbf{K})$: indeed, this means that the rank of some fixed Sylvester matrix is between two given values.

E.g. being of dimension k is a locally closed condition, or being of dimension k and of degree d is also a locally closed condition.

So our general hypothesis of fixing the Hilbert polynomial is not a priori the only natural condition that could be investigated in order to get continuity results. See e.g. [12].

Example (E) (continued) In example (E) the set $S = \mathbf{K}^2$ of parameters (a, b) is mapped in $\mathbf{K}^\Theta$ with $lst = (3, 2, 3, 4)$. The partition by subsets where the Hilbert polynomial is constant is

$$\{S_1, S_2, S_3\} = \{\{a \neq 0\}, \{a = 0, b \neq 0\}, \{a = b = 0\}\}.$$

They are indeed locally closed subsets.

2.3 Some open conditions when the Hilbert polynomial is known

We continue here using the notations of section 2.1. Moreover, since we use lemma 1.9 we assume in all this subsection that $\mathbf{K}$ has zero characteristic.

Definition 2.7 *A* deshomogenization of an algebra $\mathbf{K}[\mathbf{x}] = \mathbf{K}[x_0, \ldots, x_n]$ *is given by an invertible matrix* $F \in \mathbf{K}^{(n+1)\times(n+1)}$*: we make the linear changes of coordinates* $(y_0, \ldots, y_n) := (x_0, \ldots, x_n)F$ *and then we specialize* y_0 *to 1.*

We shall say that a deshomogenization of an algebra $\mathbf{K}[\mathbf{x}]$ *is a* good deshomogenization for an homogeneous saturated ideal $\mathcal{I}$ *if we get* $\mathcal{I}$ *by rehomogeneising. This means that the multiplication by* y_0 *is injective in* $\mathbf{K}[\mathbf{x}]/\mathcal{I}$*.*
In the same manner we say that a deshomogenization of an algebra $\mathbf{K}[\mathbf{x}]$ *is a* good deshomogenization for a system $\mathbf{f}$ of homogeneous polynomials *if it is a good deshomogenization for the corresponding saturated ideal.*

Proposition 2.8 *The property of a linear change of variables* F *for providing a good deshomogenization of a system* $\mathbf{f}^{[\gamma]}$ *is an open condition with respect to* F *and* $\mathbf{f}^{[\gamma]} \in \mathcal{HP}_{lst,H}(\mathbf{K})$*.*

Proof.
This property expresses that the product by a linear form is injective in degree D_0 (defined in 2.1). So we apply lemma 1.9 □

Example (E) (continued) Let us analyze the two cases S_1 and S_2 (S_3 is reduced to a point). When parameters are in S_1, the variety consists of at most 24 points, therefore a deshomogenization is good iff it does not send any of these points to infinity. Even in that simple case, previous result is not straightforward if we do not use the fact (proved e.g. in the companion paper [13] when the Hilbert polynomial is kept fixed) that the zero set varies continusously with respect to the parameters. When parameters are in S_2, the variety consists of the two conics Γ_1 and Γ_2, so the deshomogenization is good iff the two planes $l_1 = 0$ and $l_2 = 0$ are not sent to infinity. This is clearly an open condition.

Definition 2.9 (Noether position) *Let* $\mathcal{I}$ *be a homogeneous ideal in* $\mathbf{K}[x_0, \ldots, x_n]$*. The quotient algebra* $A = \mathbf{K}[\mathbf{x}]/\mathcal{I}$ *of dimension* m *is said in Noether position with respect to* $(x_0, \ldots, x_m)$ *iff* A *is a finite* $\mathbf{K}[x_0, \ldots, x_m]$*-module.*

Geometrically, assume that the variety $\mathbf{V}(\mathcal{I})$ defined by $\mathcal{I}$ is of dimension m. Let T be the $(n-m-1)$-projective plane inside $\mathbb{P}^n(\mathbf{C})$ defined by $\{x_1 = \ldots = x_m = 0\}$ and P be any m-dimensional projective plane not intersecting T. Let $\pi_{T,P}$ be the projection of vertex T onto P. Then Noether position means that $\mathbf{V}(\mathcal{I})$ does not intersect T. Moreover $\pi_{T,P}(\mathbf{V}(\mathcal{I})) = P$.

Proposition 2.10 *The property of a linear change of variables* F *for providing Noether position for a homogeneous ideal* $\mathbb{I}(\mathbf{f}^{[\gamma]})$ *is an open condition with respect to* F *and* $\mathbf{f}^{[\gamma]} \in \mathcal{HP}_{lst,H}(\mathbf{K})$*. In fact, we can allow* $\mathbf{f}^{[\gamma]}$ *to vary inside the Zariski locally closed set of systems giving a variety of fixed codimension* m *(see remark 2.6).*

Proof.
This property expresses that the intersection between the variety defined by $\mathbb{I}(\mathbf{f}^{[\gamma]})$ and a projective linear subspace of codimension $m-1$ defined by F is empty. So it suffices to apply theorem 2.3. □

Example (E) (continued) When parameters are in S_1, Noether position means that the deshomogenization is good. When parameters are in S_2, good deshomogenization is required but not enough. Indeed let Δ be the line corresponding to the vanishing of the last new coordinates, we have to express that Δ intersects neither the conic Γ_1 which is fixed, nor the conic Γ_2 which varies with b. This last condition means first that Δ is not in the varying plane $l_2 = 0$ and then that their intersection point is not on the fixed quadric $c_2 = 0$. So we get clearly open conditions.

Theorem 2.11 (equidimensionality and irreducibility are open conditions for fixed Hilbert polynomial) *Consider polynomial systems* $\mathbf{f}^{[\gamma]}(\mathbf{x})$ *in* $\mathcal{HP}_{lst,H}(\mathbf{K})$*, i.e. with fixed pattern lst and Hilbert polynomial* H*. Then the subset of systems giving an equidimensional scheme is* $\mathbf{K}_0$*–Zariski open in* $\mathcal{HP}_{lst,H}(\mathbf{K})$*. The same result follows for irreducible schemes.*

Proof.
Let $\mathcal{I} = \mathbb{I}(\mathbf{f}^{[\gamma]})$. By the previous proposition, we can assume that all the systems are in Noether position with respect to S with $S = \{x_1 = \ldots = x_m = 0\}$. Let $(\mathbf{y}) = (x_{m+1}, \ldots, x_n)$ and $(\mathbf{u}) = (x_0, \ldots, x_m)$.

For equidimensionality we must test that $\mathbf{K}(\mathbf{u})\mathcal{I} \cap \mathbf{K}[\mathbf{u},\mathbf{y}] = \mathcal{I}$, i.e. no element of $\mathbf{K}[\mathbf{u}]$ is a zero divisor in the quotient algebra. For irreducibility we must test that the quotient algebra has no zero divisor.

If we can give bounds a priori (i.e., if we can give bounds depending only on the format lst) on the degrees for which we have to make the tests, then we get the result by applying lemma 1.9 with restrictions of the bilinear map "product inside the quotient algebra" to suitable quotients of finite dimensional vector spaces.

Such bounds exist because there exist uniform rational algorithms for testing equidimensionality or (absolute) irreducibility. For example see [27], [24] chapter VIII sections 8, 9 or the literature about Gröbner bases. □

Notation 2.12
We denote by $\mathcal{HP}^{eqdim}_{lst,H}(\mathbf{K})$ the open subset of $\mathcal{HP}_{lst,H}(\mathbf{K})$ made of polynomial systems defining an equidimensional projective scheme.

3 First forms of continuity results

First we give elementary forms of continuity for belonging to the saturation of an homogeneous ideal or for vanishing on its zero set when the Hilbert polynomial is known.

Then we provide continuity results for parametrizations of the corresponding membership problem and Nullstellensatz. In the complex case we get global parametrizations by rational polynomials in the real part and the imaginary part of the coefficients.

3.1 Some closed conditions for continuity

In this subsection, we use the notations given in section 2.1. Moreover we fix now a new degree d and we consider the coefficient space $\mathbf{K}^{\Theta''} = \mathbf{K}^{\Theta} \times \mathbf{K}^{\theta(n,d)}$ ($\theta(n,d) = \binom{n+d}{n}$) corresponding to the data $lst'' = (n, d_1, \ldots, d_p, d)$. We denote by γ an element of the coefficient space $\mathbf{K}^{\Theta}$ (for the polynomials $f_1, \ldots, f_p$), by γ' an element of the coefficient space $\mathbf{K}^{\theta(n,d)}$ (for the polynomial g) and by $\gamma'' = (\gamma, \gamma')$ an element of the global coefficient space $\mathbf{K}^{\Theta''}$. The corresponding coefficient projective space is $\mathbb{P}^{\Theta-1}(\mathbf{K}) \times \mathbb{P}^{\theta(n,d)-1}(\mathbf{K})$.

Theorem 3.1 (1^{st} form of geometric continuity of the zero set with fixed Hilbert polynomial)
Consider polynomial systems $\mathbf{f}^{[\gamma]}(\mathbf{x})$ in $\mathcal{HP}_{lst,H}(\mathbf{K})$, i.e. with fixed pattern lst and Hilbert polynomial H. Consider also a new general polynomial g with degree d and the same number of variables. Then the condition

$$g \textit{ vanishes at the zeroes of } \mathbf{f} \textit{ in } \mathbb{P}^n(\mathbf{C})$$

is a closed one over $\mathcal{HP}_{lst,H}(\mathbf{K})$. More precisely, the subset $\mathcal{HP}^{geom\in}_{lst,d,H}(\mathbf{K})$ of $\mathcal{HP}_{lst,H}(\mathbf{K}) \times \mathbb{P}^{\theta(n,d)-1}(\mathbf{K})$ defined by this condition is $\mathbf{K}_0$–Zariski closed.

Proof.
By using the bounds in 1.4, the condition " g vanishes at the zeroes of $\mathbf{f}$ in $\mathbb{P}^n(\mathbf{C})$ " is equivalent to: " $g^N \in \mathbb{I}(\mathbf{f})$ " for some explicit exponent $N = N_3(n, \max(d, d_1, \ldots, d_p))$. This is equivalent to: the Sylvester matrix in degree $D = Nd$ of the polynomial system $\mathbf{f}(\mathbf{x})$ has $g^N(\mathbf{x})$ in its image. So apply lemma 1.6 a). □

Corollary 3.2 *Same hypotheses as in theorem 3.1. Let us denote $S = \mathcal{HP}_{lst,H}(\mathbf{C})$,*

$$V = \left\{(\gamma, \mathbf{x}) : \mathbf{f}^{[\gamma]}(\mathbf{x}) = 0\right\} \subset S \times \mathbb{P}^n(\mathbf{C}), \qquad V_\gamma = \left\{\mathbf{x} : \mathbf{f}^{[\gamma]}(\mathbf{x}) = 0\right\} \subset \mathbb{P}^n(\mathbf{C})$$

and $\pi_V = V \to S$ the restriction of the canonical projection $S \times \mathbb{P}^n(\mathbf{C}) \to S$. Then π_V is a Zariski open mapping.

Proof.
Let $g(\gamma, \mathbf{x})$ be a polynomial which is $\mathbf{x}$-homogeneous and γ-homogeneous. The condition $g(\gamma, \mathbf{x}) \neq 0$

defines a basic Zariski open set on $\mathbb{P}^{\Theta-1}(\mathbf{C}) \times \mathbb{P}^n(\mathbf{C})$ whose intersection with V is a basic open set U. We have to show that $\pi_V(U)$ is Zariski open in S. Consider

$$F' = \{\, \gamma \in S \;:\; g(\gamma, \bullet) = 0 \text{ on } V_\gamma \}.$$

If the $\mathbf{x}$-degree of g is equal to d call F the closed subset $\mathcal{HP}^{geom\in}_{lst,d,H}(\mathbf{C})$ of $S \times \mathbb{P}^{\theta(n,d)-1}(\mathbf{C})$ given by theorem 3.1. Then F' is the inverse image of F by the mapping

$$\gamma \longmapsto (\gamma, g(\gamma, \bullet)) \qquad \mathbb{P}^{\Theta-1}(\mathbf{C}) \longrightarrow \mathbb{P}^{\Theta-1}(\mathbf{C}) \times \mathbb{P}^{\theta(n,d)-1}(\mathbf{C})$$

So F' is Zariski closed, i.e.,

$$\{\, \gamma \in S \;:\; \exists \mathbf{x} \in V_\gamma \;\; g(\gamma, \mathbf{x}) \neq 0 \,\}$$

is Zariski open. And this set is exactly $\pi_V(U)$. □

Theorem 3.3 (1st form of algebraic continuity of the zero set with fixed Hilbert polynomial) *Consider polynomial systems* $\mathbf{f}^{[\gamma]}(\mathbf{x})$ *in* $\mathcal{HP}_{lst,H}(\mathbf{K})$*, i.e. with fixed pattern lst and Hilbert polynomial H. Consider also a new general polynomial g with degree d and the same number of variables. Then the condition*

$$g \text{ belongs to the saturation of } \mathbb{I}(\mathbf{f})$$

is a closed one over $\mathcal{HP}_{lst,H}(\mathbf{K})$*. More precisely, the subset* $\mathcal{HP}^{alg\in}_{lst,d,H}(\mathbf{K})$ *of* $\mathcal{HP}_{lst,H}(\mathbf{K}) \times \mathbb{P}^{\theta(n,d)-1}(\mathbf{K})$ *defined by this condition is* $\mathbf{K}_0$*–Zariski closed.*

Proof.
From the bounds in 1.4 the condition " g belongs to the saturation of $\mathbb{I}(\mathbf{f})$ " is equivalent to: " for $j \in \{0, \ldots, n\}$ $g x_j^N$ belongs to $\mathbb{I}(\mathbf{f})$ " for some explicit $N = N_1(n, \max(d_1, \ldots, d_p))$. So apply lemma 1.6 a). □

3.2 Continuous rational parametrizations

In this section we study the possibility of giving local and/or global continuous parametrizations for some homogeneous Nullstellensätze and membership equalities when the Hilbert polynomial is kept fixed.

We use notations of section 2.1. Moreover, when dealing with global parametrizations (theorems 3.6 to 3.10), $\mathbf{K}$ is an ordered field contained in a real closed field $\mathbf{R}$ and $\mathbf{C} = \mathbf{R}[\sqrt{-1}]$.

Preliminary remarks
In the real case, global continuous parametrizations for the real Nullstellensatz, the solution to seventeenth Hilbert problem and other forms of Positivstellensatz have been obtained by C. Delzell, P. Scowcroft and the two last authors in [4], [5], [29], [7], [6], [14], [15]. The technique appearing in [7] is simple and straightforward. We shall use a similar technique here for the complex case in the last paragraph of this section.

From a remark of C. Delzell, we know that any continuous parametrization of any variant of a Nullstellensatz which is valid inside some subset S of some coefficient space M, is extendible to a bigger locally closed subset S', under very weak assumptions. Indeed, let $f_1(\mathbf{c}, \mathbf{x}), \ldots, f_p(\mathbf{c}, \mathbf{x})$ be p polynomials in $\mathbf{x}$ with coefficients $\mathbf{c} \in M$, and assume that the incompatibility in $\mathbf{x}$ of some system of conditions on $f_1(\mathbf{c}, \mathbf{x}), \ldots, f_p(\mathbf{c}, \mathbf{x})$ is certified by a special kind of algebraic identity (a Truc-stellensatz), given continuously with respect to $\mathbf{c} \in S$. I.e. the algebraic identity has a fixed form and the coefficients appearing in the algebraic identity are given by continuous functions of $\mathbf{c} \in S$. We assume that these functions are defined and continuous over all M (or that they can be continuously extended to all M). Being an algebraic identity remains true on the closure $\overline{S}$ of S. Some conditions on the coefficients are moreover needed (e.g. some coefficient is nonzero, or nonnegative and so on) in order to get a good certificate. In practice, these conditions are always open or closed conditions, so they define a locally closed subset $S' \subseteq \overline{S}$ which contains S.

So, in the following, we restrict systematically ourselves to the locally closed case.

Let us give here a very simple example of a non continuously parametrizable complex Nullstellensatz (a similar one was given by Kreisel). We consider two univariate polynomials of degree 1:

$$f(x) = ax + b \quad \text{and} \quad g(x) = cx + d.$$

The implication

$$\forall x\ (\ f(x) = 0 \ \ \Rightarrow \ \ g(x) = 0\ )$$

is equivalent to

$$(a \neq 0,\ ad - bc = 0) \quad \text{or} \quad (a = 0,\ b \neq 0) \quad \text{or} \quad (a = 0,\ b = 0,\ c = 0,\ d = 0),$$

which corresponds to a non locally closed set for (a, b, c, d). In the first case, a Nullstellensatz is given by $ag = cf$. In the second case by $bg = (cx + d)f$. In the third case by $g = f$. And there exist no general identity $h(a, b, c, d, x)\ g^m = k(a, b, c, d, x)\ f$ with h and k polynomials in x depending continuously on (a, b, c, d) and $h(a, b, c, d, x)$ non identically zero for all (a, b, c, d).

Applying previous results

In this paragraph we obtain our results by immediate application of theorems 3.1 and 3.3 and results of sections 1.5 and 1.6.

Theorem 3.4 (local continuous parametrization of Nullstellensatz when Hilbert polynomial is known) *Consider polynomial systems* $\mathbf{f}^{[\gamma]}(\mathbf{x})$ *in* $\mathcal{HP}_{lst,H}(\mathbf{K})$*, i.e. with fixed pattern lst and Hilbert polynomial* H*. Consider also a new general polynomial* g *with degree* d *and the same number of variables. Then there is a finite* $\mathbf{K}_0$*–Zariski open cover* (V_i) *of* $\mathcal{HP}_{lst,H}(\mathbf{K})$ *such that over each* V_i *there is a uniform way of writing the corresponding Nullstellensatz using an integer* N *and polynomials* $q_i \in \mathbf{K}_0[c_1, \ldots, c_m]$, $a_{i1}, \ldots, a_{ip} \in \mathbf{K}_0[c_1, \ldots, c_{m''}, \mathbf{x}]$*:*

$$q_i(\gamma)\ g(\gamma', \mathbf{x})^N \ = \ a_{i1}(\gamma'', \mathbf{x})\ f_1(\gamma, \mathbf{x}) \ + \ \cdots \ + \ a_{ip}(\gamma'', \mathbf{x})\ f_p(\gamma, \mathbf{x})$$

(with $q_i(\gamma)$ *nowhere vanishing on* V_i*) when* g *vanishes at the zeroes of* $\mathbf{f}$ *in* $\mathbb{P}^n(\mathbf{C})$*.*

Proof.

Same reasoning as in theorem 3.1 and apply lemma 1.6 b). □

Example (E) (continued) When parameters are in S_1, we consider a parametrized Nullstellensatz for the vanishing of the varying polynomial $g = l_1 c_1 c_3$. We find

$$ag = -c_1^2 f_1 + x c_1 f_2 - x^2 f_3.$$

In this case, the situation is simple and we get directly a globally parametrized formula, which is a rational function of the parameter, the denominator being nonzero on S_1. However in more complicated situations, it should be necessary to work more and to glue local formulas.

Theorem 3.5 (local continuous parametrization for belonging to the saturation of an homogeneous ideal when the Hilbert polynomial is known) *Consider polynomial systems* $\mathbf{f}^{[\gamma]}(\mathbf{x})$ *in* $\mathcal{HP}_{lst,H}(\mathbf{K})$*, i.e. with fixed pattern lst and Hilbert polynomial* H*. Consider also a new general polynomial* g *with degree* d *and the same number of variables. Then there exists a finite* $\mathbf{K}_0$*–Zariski open cover* (W_i) *of* $\mathcal{HP}_{lst,H}(\mathbf{K})$ *such that over each* W_i *there is a uniform way of writing the corresponding fact using an integer* N *and polynomials* $q_i \in \mathbf{K}_0[c_1, \ldots, \mathbf{c}_m]$, $a_{i,j,1}, \ldots, a_{i,j,p} \in \mathbf{K}_0[c_1, \ldots, \mathbf{c}_{m''}, \mathbf{x}]$ *(*$j = 0, \ldots, n$*):*

$$q_i(\gamma)\ g(\gamma', \mathbf{x})\ x_j^N \ = \ a_{i,j,1}(\gamma'', \mathbf{x})\ f_1(\gamma, \mathbf{x}) \ + \ \cdots \ + \ a_{i,j,p}(\gamma'', \mathbf{x})\ f_p(\gamma, \mathbf{x})$$

(with $q_i(\gamma)$ *nowhere vanishing on* W_i*) when* g *belongs to the saturation of* $\mathbb{I}(\mathbf{f})$*.*

Proof.

Same reasoning as in theorem 3.3 and apply lemma 1.6 b). □

Now $\mathbf{K}$ is an ordered field contained in a real closed field $\mathbf{R}$ and $\mathbf{C} = \mathbf{R}[\sqrt{-1}]$. We use definition 1.12.

Theorem 3.6 (global real parametrization of the homogeneous complex Nullstellensatz when the Hilbert polynomial is known) *Consider a pattern (lst, d) and a Hilbert polynomial H. Using the previous notations, we claim that there exists a global parametrization of the corresponding homogeneous Nullstellensatz by $\mathbb{Q}$–real polynomials: there exist a nonnegative integer N, a $\mathbb{Q}$–real polynomial $q \in \mathbb{Q}\lfloor \mathbf{c} \rceil$ positive on $\mathcal{HP}^{geom\in}_{lst,d,H}(\mathbf{C})$, and $a_k \in \mathbb{Q}\lfloor \mathbf{c} \rceil[\mathbf{x}]$ $(k \in \{1, \ldots, p\})$ such that if $\gamma'' = (\gamma, \gamma') \in \mathcal{HP}^{geom\in}_{lst,d,H}(\mathbf{C})$ then we have the following $\mathbf{x}$–algebraic identity*

$$q(\gamma)\ g(\gamma', \mathbf{x})^N = \sum_{k=1}^{p} a_k(\gamma'', \mathbf{x})\ f_k(\gamma, \mathbf{x}).$$

Proof.
Same reasoning as in theorem 3.1 and apply lemma 1.15. □

Theorem 3.7 (global continuous parametrization of identities showing that a polynomial belongs to the saturation of a varying ideal when the Hilbert polynomial is known) *Consider a pattern (lst, d) and a Hilbert polynomial H. Using the previous notations, we claim that there exists a global continuous parametrization over $\mathcal{HP}_{lst,H}(\mathbf{C})$ for g being in the saturation of the ideal $\mathbb{I}(\mathbf{f})$ given in the following form $(0 \le j \le n)$ by $\mathbf{x}$–homogeneous equalities*

$$q(\gamma)\ g(\gamma', \mathbf{x})\ x_j^N \ =\ a_{j,1}(\gamma'', \mathbf{x})\ f_1(\gamma, \mathbf{x})\ +\ \cdots\ +\ a_{j,p}(\gamma'', \mathbf{x})\ f_p(\gamma, \mathbf{x})$$

Here N is an integer depending on lst, $q(\gamma)$ and $a_{j,k}(\gamma'', \mathbf{x})$ are $\mathbb{Q}$–real polynomials and $q(\gamma)$ is positive on $\mathcal{HP}_{lst,H}(\mathbf{C})$.

Proof.
Same reasoning as in theorem 3.3 and apply lemma 1.15. □

Remark that for $\gamma \in \mathcal{HP}_{lst,H}(\mathbf{C})$, g belongs to the saturation of $\mathbb{I}(\mathbf{f})$ if and only if the above equalities are valid.

As particular cases of theorem 3.6 we get.

Theorem 3.8 (global parametrization of the Weak Homogeneous Nullstellensatz)
Using the previous notations, we define $\mathrm{W}_{lst}(\mathbf{C})$ as the set of all the $\gamma = (\gamma_1, \ldots, \gamma_\Theta) \in \mathbf{C}^\Theta$ such that the system of polynomial equations

$$f_1(\gamma, \mathbf{x}) = 0,\ \ldots,\ f_p(\gamma, \mathbf{x}) = 0$$

has no solutions except $(0, \ldots, 0)$.
Then $\mathrm{W}_{lst}(\mathbf{C})$ is $\mathbb{Q}$–Zariski open and there exists a global parametrization of the corresponding Nullstellensatz by $\mathbb{Q}$–real polynomials: there exist a nonnegative integer N, a $\mathbb{Q}$–real polynomial $q \in \mathbb{Q}\lfloor \mathbf{c} \rceil$ positive on $\mathrm{W}_{lst}(\mathbf{C})$, and $a_k^{(j)} \in \mathbb{Q}\lfloor \mathbf{c} \rceil[\mathbf{x}]$ $(k \in \{1, \ldots, p\}$, $j \in \{0, \ldots, n\})$ such that if $\gamma \in \mathrm{W}_{lst}(\mathbf{C})$ then we have the following $\mathbf{x}$–algebraic identity

$$q(\gamma)\ x_j^N = \sum_{k=1}^{p} a_k^{(j)}(\gamma, \mathbf{x})\ f_k(\gamma, \mathbf{x}).$$

Another method
Here we give a method inspired by [7, 14, 15]. We get perhaps a more general result than in the previous paragraph. Remark that in the previous paragraph we examined only the projective case, but the same techniques work in the affine case.

Theorem 3.9 (global real parametrization of some instances of the complex Nullstellensatz). *Let $f_1, \ldots, f_p, g$ be polynomials in $\mathbf{K}[c_1, \ldots, c_m, x_1, \ldots . x_n]$ and S be a $\mathbf{K}$–Zariski locally closed set in the parameter space $\mathbf{C}^m$ such that:*

$$\gamma \in S \implies [\ \forall \xi \in \mathbf{C}^n \quad (\ (f_1(\gamma, \xi) = 0, \ldots, f_p(\gamma, \xi) = 0)\ \Rightarrow\ g(\gamma, \xi) = 0\ )\ ]$$

Then there exists a continuous parametrization on S for the Nullstellensatz corresponding to the implication:

$$\forall \xi \in \mathbb{C}^n \quad (f_1(\gamma,\xi) = 0, \ldots, f_p(\gamma,\xi) = 0) \ \Rightarrow\ g(\gamma,\xi) = 0$$

More precisely, there exist a nonnegative integer N, a $\mathbf{K}$–real polynomial $q \in \mathbf{K}\lfloor\mathbf{c}\rceil$ positive on S, and $v_j \in \mathbf{K}\lfloor\mathbf{c}\rceil[\mathbf{x}]$ ($j \in \{1,\ldots,p\}$) such that if $\gamma \in S$ then we have the following $\mathbf{x}$–algebraic identity

$$q(\gamma) g(\gamma,\mathbf{x})^N = \sum_{j=1}^{p} v_j(\gamma,\mathbf{x}) f_j(\gamma,\mathbf{x}).$$

Proof.
The technique is very similar to the one used in [7] for parametrizing the real Positivstellensatz. Let $S = F \cap U$ where F is a $\mathbf{K}$–Zariski closed set defined by $(\varphi_h(\gamma) = 0)_{h=1,\ldots,r}$ and U is a $\mathbf{K}$-Zariski open set, union of basic open sets defined by $\psi_k(\gamma) \neq 0$ for $k = 1,\ldots,q$. For each $k = 1,\ldots,q$, consider the following incompatible system:

$$\varphi_1(\gamma) = 0, \ldots, \varphi_r(\gamma) = 0,\ f_1(\gamma,\xi) = 0, \ldots, f_p(\gamma,\xi) = 0,\ \psi_k(\gamma) \neq 0, g(\gamma,\xi) \neq 0.$$

Hilbert's Nullstellensatz gives an algebraic identity E_k in $\mathbf{K}[\mathbf{c},\mathbf{x}]$

$$(\psi_k(\mathbf{c}) g(\mathbf{c},\mathbf{x}))^{N_k} = \varphi_1(\mathbf{c}) a_{k,1}(\mathbf{c},\mathbf{x}) + \cdots + \varphi_r(\mathbf{c}) a_{k,r}(\mathbf{c},\mathbf{x}) + f_1(\mathbf{c},\mathbf{x}) b_{k,1}(\mathbf{c},\mathbf{x}) + \cdots + f_p(\mathbf{c},\mathbf{x}) b_{k,p}(\mathbf{c},\mathbf{x})$$

We may assume w.l.o.g. that all exponents N_k are equal to $N \in \mathbb{N}$. We Multiply each E_k by $\overline{\psi_k(\mathbf{c})}^N$ and add these algebraic identities in $\mathbf{K}\lfloor\mathbf{c}\rceil[\mathbf{x}]$:

$$\left(\sum_k |\psi_k(\mathbf{c})|^{2N}\right) g(\mathbf{c},\mathbf{x})^N = \left(\sum_k \overline{\psi_k(\mathbf{c})}^N a_{k,1}(\mathbf{c},\mathbf{x})\right) \varphi_1(\mathbf{c}) + \cdots + \left(\sum_k \overline{\psi_k(\mathbf{c})}^N a_{k,r}(\mathbf{c},\mathbf{x})\right) \varphi_r(\mathbf{c})$$
$$+ \left(\sum_k \overline{\psi_k(\mathbf{c})}^N b_{k,1}(\mathbf{c},\mathbf{x})\right) f_1(\mathbf{c},\mathbf{x}) + \cdots + \left(\sum_k \overline{\psi_k(\mathbf{c})}^N b_{k,p}(\mathbf{c},\mathbf{x})\right) f_p(\mathbf{c},\mathbf{x}).$$

For $\gamma \in S$, we get an algebraic identity in $\mathbf{K}[\mathbf{x}]$ parametrized by $\mathbf{K}$–real polynomials

$$\left(\sum_k |\psi_k(\gamma)|^{2N}\right) g(\gamma,\mathbf{x})^N = \left(\sum_k \overline{\psi_k(\gamma)}^N b_{k,1}(\gamma,\mathbf{x})\right) f_1(\gamma,\mathbf{x}) + \cdots + \left(\sum_k \overline{\psi_k(\gamma)}^N b_{k,p}(\gamma,\mathbf{x})\right) f_p(\gamma,\mathbf{x})$$

i.e.,

$$q(\gamma) g(\gamma,\mathbf{x})^N = v_1(\gamma,\mathbf{x}) f_1(\gamma,\mathbf{x}) + \cdots + v_p(\gamma,\mathbf{x}) f_p(\gamma,\mathbf{x})$$

with $q(\gamma)$ everywhere positive on S. □

Remark that, in the previous proof, $q(\gamma) > 0$ is clearly a defining inequation of the Zariski open set U. An immediate corollary of the previous theorem is:

Theorem 3.10 *Let $\mathbf{f} = (f_1,\ldots,f_p)$ and $\mathbf{g} = (g_1,\ldots,g_r)$ be two lists of polynomials in $\mathbf{K}[\mathbf{c},\mathbf{x}]$. Let $\mathbf{S}_{\mathbf{f},\mathbf{g}}(\mathbf{C})$ be the $\mathbf{K}$–constructible set in the parameter space $\mathbf{C}^m$ defined by:*

$$\gamma \in \mathbf{S}_{\mathbf{f},\mathbf{g}}(\mathbf{C}) \Longleftrightarrow \forall \xi \in \mathbf{C}^n\ [((f_1(\gamma,\xi) = 0, \ldots, f_p(\gamma,\xi) = 0) \ \Rightarrow\ (g_1(\gamma,\xi) = 0, \ldots, g_r(\gamma,\xi) = 0))]$$

If $\mathbf{S}_{\mathbf{f},\mathbf{g}}$ is a Zariski locally closed set in $\mathbf{C}^m$ then there exists a global continuous parametrization on $\mathbf{S}_{\mathbf{f},\mathbf{g}}(\mathbf{C})$ for the Nullstellensätze corresponding to the implications:

$$\forall \xi \in \mathbf{C}^n \quad (f_1(\gamma,\xi) = 0, \ldots, f_p(\gamma,\xi) = 0) \ \Rightarrow\ g_i(\gamma,\xi) = 0$$

More precisely, there exist a nonnegative integer N, a $\mathbf{K}$–real polynomial $q \in \mathbf{K}\lfloor\mathbf{c}\rceil$ positive on $\mathbf{S}_{\mathbf{f},\mathbf{g}}(\mathbf{C})$, and $a_j^{(i)} \in \mathbf{K}\lfloor\mathbf{c}\rceil[\mathbf{x}]$ ($j \in \{1,\ldots,p\}$, $i \in \{1,\ldots,r\}$) such that if $\gamma \in \mathbf{S}_{\mathbf{f},\mathbf{g}}(\mathbf{C})$ then we have the following $\mathbf{x}$–algebraic identity

$$q(\gamma)\ g_i(\gamma,\mathbf{x})^N = \sum_{j=1}^{p} a_j^{(i)}(\gamma,\mathbf{x}) f_j(\gamma,\mathbf{x}).$$

Corollary 3.11 *The* $\mathbf{x}$*–homogeneous versions of theorem 3.9 and 3.10 can be easily obtained by merely considering the appropriate* $\mathbf{x}$*–homogeneous part of the polynomials in the algebraic identity. So applying theorem 3.1 we get another proof for theorems 3.6 and 3.8.*

As an application, next corollary shows how previous results give a constructive version of complex Nullstellensatz when the coefficients of the involved polynomials are Cauchy–complex numbers, i.e., their real and imaginary parts are given by rational approximations.

Corollary 3.12 *Each global continuous parametrization of any instance of the complex Nullstellensatz provides a constructive version of the same instance of the complex Nullstellensatz when dealing with polynomials whose coefficients are Cauchy–complex numbers.*

Indeed as the exponents are uniformly bounded and the coefficients vary continuously (and are clearly non zero when needed), one can compute approximations as precise as desired for any set of parameters.

Conclusion

In this paper, we studied continuity properties of the solutions of parametrized systems of homogeneous polynomial equations $\mathbf{f}^{[\gamma]}$ from a topological point of view.

First, we proved that having a given Hilbert polynomial of the corresponding ideal is a locally closed condition. Then, when the Hilbert polynomial of the corresponding ideal is kept fixed, we proved the following assertions:

- The property of a linear change of variables for providing a good desomogeneization or Noether position of $\mathbb{I}(\mathbf{f}^{[\gamma]})$ is an open condition with respect to the coefficients of the change of variable and of $\mathbf{f}^{[\gamma]}$.
- The properties of a scheme, being equidimensional or irreducible, are open.
- The conditions for an homogeneous polynomial g of fixed degree in the same variable:

  $$g \text{ vanishes at the zeroes of } \mathbf{f} \text{ in } \mathbb{P}^n(\mathbf{C})$$

  and

  $$g \text{ belongs to the saturation of } \mathbb{I}(\mathbf{f})$$

  are a closed condition with respect to the coefficients of g and of $\mathbf{f}$.
- There exist local and/or global continuous parametrizations for some homogeneous Nullstellensätze and membership equalities. More precisely the dependency on the parameters in the corresponding algebraic identities is expressed via $\mathbb{Q}$–real polynomials. See theorems 3.1 to 3.10.

  This implies useful constructive versions of complex Nullstellensatz when dealing with polynomials given by Cauchy-complex numbers.

In the companion paper [13] we shall also study uniform continuity results w.r.t. Hausdorff distance. So we will generalize previous results obtained for the univariate case; see e.g. [26], [10] or [9].

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