The Gröbner Ring Conjecture in One Variable

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Abstract

We prove that a valuation domain $\mathbf{V}$ has Krull dimension ≤ 1 if and only if for every finitely generated ideal I of $\mathbf{V}[X]$ the ideal generated by the leading terms of elements of I is also finitely generated. This proves the Gröbner ring conjecture in one variable. The proof we give is both simple and constructive. The same result is valid for semihereditary rings.

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Introduction

Recall that according to [9] a ring $\mathbf{R}$ is said to be *Gröbner* if for every $n \in \mathbb{N}$ and every finitely generated ideal I of $\mathbf{R}[X_1, \dots, X_n]$, fixing a monomial order on $\mathbf{R}[X_1, \dots, X_n]$, the ideal $\mathrm{LT}(I)$ generated by the leading terms of the elements of I is finitely generated. The *Gröbner ring conjecture* [9] says that a valuation domain is Gröbner if and only if its Krull dimension is ≤ 1.

Recall further that a *valuation domain* is a domain $\mathbf{V}$ such that for any a, b in $\mathbf{V}$ either a divides b or b divides a. This means that every finitely generated ideal is principal, and thus a free module. Moreover, a ring is called *semihereditary* whenever every finitely generated ideal is a projective module.

We prove (Theorem 4) that a valuation domain $\mathbf{V}$ satisfies the property "for any finitely generated ideal I of $\mathbf{V}[X]$ the ideal $\mathrm{LT}(I)$ is finitely generated" if and only if its Krull dimension is ≤ 1. This proves the Gröbner ring conjecture in one variable, and also gives the first example of a class of non-Noetherian rings satisfying the property above. The proof we give is both simple and constructive. The same result is valid for semihereditary rings (Corollary 5).

The paper is written in Bishop–style constructive mathematics (see [8] for basic algebra).

1 A simple result about coherent rings

Let $\mathbf{A}$ be an arbitrary commutative ring.

For a polynomial $g = \sum_j a_j X^j \in \mathbf{A}[X]$, we set $\mathrm{coeff}_X(f, k) := a_k$. If the degree of g is known, we denote by $\mathrm{LT}(g)$, $\mathrm{LM}(g)$, $\mathrm{LC}(g)$ respectively the leading term of g, its leading monomial and its leading coefficient.

We denote by $\mathbf{A}[X]_k$ the free submodule of rank $k+1$ of $\mathbf{A}[X]$ generated by $1, X, \dots, X^k$. If I is an ideal of $\mathbf{A}[X]$ we denote by I_k the submodule $I \cap \mathbf{A}[X]_k$. If $\mathbf{A}$ is discrete we denote by $\mathrm{LT}(I)$ the ideal $\langle \mathrm{LT}(f) : f \in I \rangle$.

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If the ring is not known to be discrete, for $f \in \mathbf{A}[X]$, $\langle \mathrm{LT}(f) \rangle$ denotes the ideal generated by the terms $a_k X^k$ of f for all k s.t. $\mathrm{coeff}(f, \ell) = 0$ for $\ell > k$. And for a subset $E \subseteq \mathbf{A}[X]$, $\mathrm{LT}(E)$ denotes the ideal $\sum_{f \in E} \langle \mathrm{LT}(f) \rangle$.

In this section we don't assume $\mathbf{A}$ to be a discrete ring.

Proposition 1 *Let $I = \langle f, f_1, \ldots, f_s \rangle_{\mathbf{A}[X]}$ be a finitely generated ideal of $\mathbf{A}[X]$, with f monic of degree n. Then*

1. *I_{n-1} is a finitely generated $\mathbf{A}$-module,*
2. *$I = \langle I_{n-1} \rangle_{\mathbf{A}[X]} + \langle f \rangle_{\mathbf{A}[X]} = I_{n-1} \oplus \langle f \rangle_{\mathbf{A}[X]}$,*
3. *$\mathrm{LT}(I) = \mathrm{LT}(I_{n-1}) + \langle X^n \rangle_{\mathbf{A}[X]}$.*

Proof. Let $\mathbf{B} = \mathbf{A}[X]/\langle f \rangle$ be the quotient algebra, which is a free $\mathbf{A}$-module with basis $1, x, \ldots, x^{n-1}$ ($x = \overline{X}$ is the class of X modulo f), let $\psi : \mathbf{B}^s \to \mathbf{B}$ be the *generalized Sylvester map*

$$(\overline{g_1}, \ldots, \overline{g_s}) \mapsto \sum_{i=1}^{s} \overline{g_i}\, \overline{f_i}.$$

Then clearly $I_{n-1}\mathbf{B}$ is generated by the image of ψ, which is the module generated by all the $x^k \overline{f_i}$ with $0 \le k < n, 1 \le i \le s$.
In matrix form we get the *generalized Sylvester matrix* associated to the polynomials $f, f_1, \ldots, f_s$ denoted by $\mathrm{Syl}_X(f, f_1, \ldots, f_s)$ which is the matrix with the following columns:

$$\mathrm{Rem}(f_1, f), \ldots, \mathrm{Rem}(f_s, f), \mathrm{Rem}(X f_1, f), \ldots, \mathrm{Rem}(X f_s, f), \ldots, \mathrm{Rem}(X^{n-1} f_1, f), \ldots, \mathrm{Rem}(X^{n-1} f_s, f)$$

(where $\mathrm{Rem}(g, f)$ denotes the remainder of the division of g by f) in the basis $(X^{n-1}, \ldots, X, 1)$. And I_{n-1} is the module generated by the columns of $\mathrm{Syl}_X(f, f_1, \ldots, f_s)$. □

Example 2 *If*

$$f(X) = X^3 + 3X^2 + 4,$$
$$f_1(X) = 4X^2 + 5X + 3, \quad f_2(X) = -3X^2 + 2X + 3, \quad f_3(X) = 2X^2 - X + 7,$$

then

$$\mathrm{Syl}_X(f, f_1, f_2, f_3) = \begin{pmatrix} 4 & -3 & 2 & -7 & 11 & -7 & 24 & -30 & 28 \\ 5 & 2 & -1 & 3 & 3 & 7 & -16 & 12 & -8 \\ 3 & 3 & 7 & -16 & 12 & -8 & 28 & -44 & 28 \end{pmatrix}.$$

Theorem 3 *Let $\mathbf{A}$ be a coherent ring and $I = \langle f, f_1, \ldots, f_s \rangle$ a finitely generated ideal of $\mathbf{A}[X]$, with f monic. Then*

1. *the elimination ideal $I_0 = I \cap \mathbf{A}$,*
2. *the elimination modules $I_k = I \cap \mathbf{A}[X]_k$, and*
3. *the leading ideal $\mathrm{LT}(I)$*

are finitely generated.

Proof. Let $\pi_k : \mathbf{A}[X]_k \to \mathbf{A}$ be the coordinate form $f \mapsto \mathrm{coeff}(f, k)$. We know that I_{n-1} is a finitely generated module. For $k \ge n$ the module $I_k = I_{n-1} \oplus f(\mathbf{A} + X\mathbf{A} + \ldots + X^{k-n}\mathbf{A})$ is finitely generated. For $k < n-1$ the module I_k is finitely generated because $I_k = I_{n-1} \cap \mathbf{A}[X]_k$, and these two modules are finitely generated submodules of the module $\mathbf{A}[X]_{n-1}$, which is isomorphic to $\mathbf{A}^n$, hence coherent. So I_k and $\pi_k(I_k)$ are finitely generated $\mathbf{A}$-modules. Thus the leading ideal

$$\mathrm{LT}(I) = \pi_0(I_0) + \pi_1(I_1)\langle X \rangle + \cdots + \pi_{n-1}(I_{n-1})\langle X^{n-1} \rangle + \langle X^n \rangle$$

is finitely generated. □

Let us describe with more details a computation corresponding to the above proof. We assume that $\deg(f) = 5$ and we want to know I_2 and the ideal generated by the terms of degree 2 for polynomials in I_2, that is $\pi_2(I_2) \cdot \langle X^2 \rangle$, where $\pi_2 : I_2 \to \mathbf{A}$ is the coordinate form $f \mapsto \text{coeff}(f, 2)$. Suppose further that the generalized Sylvester matrix has the following pattern

$$\begin{matrix} X^4 \\ X^3 \\ X^2 \\ X \\ 1 \end{matrix} \quad \begin{pmatrix} c_1 & c_2 & c_3 & c_4 & \cdots\cdots\cdots\cdots \\ b_1 & b_2 & b_3 & b_4 & \cdots\cdots\cdots\cdots \\ a_1 & a_2 & a_3 & a_4 & \cdots\cdots\cdots\cdots \\ v_1 & v_2 & v_3 & v_4 & \cdots\cdots\cdots\cdots \\ u_1 & u_2 & u_3 & u_4 & \cdots\cdots\cdots\cdots \end{pmatrix}$$

with ℓ columns. We have

$$\pi_2(I_2) = \left\{ \sum\nolimits_{i=1}^{\ell} \alpha_i a_i \right\}$$

for all $(\alpha_1, , \ldots, \alpha_\ell)$ that are linear dependence relations for the family

$$U = \left(\begin{pmatrix} c_1 \\ b_1 \end{pmatrix}, \ldots, \begin{pmatrix} c_\ell \\ b_\ell \end{pmatrix} \right) \in (\mathbf{A}^2)^\ell.$$

Similarly

$$I_2 = \left\{ \sum\nolimits_{i=1}^{\ell} \alpha_i (u_i + v_i X + a_i X^2) \right\}$$

for the same $(\alpha_1, , \ldots, \alpha_\ell)$'s.
Since $\mathbf{A}$ is a coherent ring, $\mathbf{A}^2$ is a coherent $\mathbf{A}$-module and the module of relations for U is finitely generated.

2 The Gröbner Ring Conjecture

Recall that a ring $\mathbf{R}$ has Krull dimension ≤ 1 if and only if

$$\forall a, b \in \mathbf{R}, \;\; \exists n \in \mathbb{N}, \;\; \exists\, x, y \in \mathbf{R} \;\;\; | \;\;\; a^n(b^n(1 + xb) + ya) = 0. \tag{1}$$

This is a constructive substitute for the classical abstract definition [1, 2, 5, 7]. For a valuation domain, it is easy to see that (1) amounts to the fact that the valuation group is archimedean.

Recall that a valuation domain $\mathbf{V}$ has dimension ≤ 1 if and only if $\mathbf{V}\langle X \rangle$ (the localization of $\mathbf{V}[X]$ at monic polynomials) is a Bezout domain (see [7] for a constructive proof).

The following is the main result of this paper.

Theorem 4 *For a valuation domain $\mathbf{V}$, the following assertions are equivalent:*

1. *For any finitely generated ideal I of $\mathbf{V}[X]$, the leading terms ideal* $\mathrm{LT}(I)$ *is also finitely generated.*
2. *If J is a finitely generated ideal of $\mathbf{V}[X]$, then $J \cap \mathbf{V}$ is a principal ideal of $\mathbf{V}$.*
3. $\dim \mathbf{V} \leq 1$.

Proof. The implications "*1.* $\Rightarrow$ *2.* $\Rightarrow$ *3.*" are proved in [4] (see proof of Theorem 11).
"*3.* $\Rightarrow$ *1.*" Let I be a finitely generated nonzero ideal of $\mathbf{V}[X]$, say $I = \langle f_1, \ldots, f_s \rangle$. Denoting by $\mathbf{K}$ the quotient field of $\mathbf{V}$ and setting $\Delta := \gcd(f_1, \ldots, f_s)$ in $\mathbf{K}[X]$, we have $I = \langle f_1, \ldots, f_s \rangle = \langle \Delta\, h_1, \ldots, \Delta\, h_s \rangle$ for some coprime polynomials $h_1, \ldots, h_s \in \mathbf{K}[X]$. Replacing I by $\alpha\, I$ for an appropriate $\alpha \in \mathbf{V} \setminus \{0\}$, we may suppose that $\Delta, h_1, \ldots, h_s \in \mathbf{V}[X]$. As $\mathbf{V}$ is a valuation domain there is one coefficient a of one of the h_i's which divides all the others. Thus, one can write $I = a\,\Delta\,\langle g_1, \ldots, g_s \rangle$ where $\Delta, g_1, \ldots, g_s \in \mathbf{V}[X]$, $\gcd(g_1, \ldots, g_s) = 1$ in $\mathbf{K}[X]$ and at least one of the g_i's is primitive.

In particular, it follows that $\gcd(g_1, \ldots, g_s) = 1$ in $\mathbf{V}[X]$. As $\mathbf{V}\langle X\rangle$ is a Bezout domain, the ideal $J = \langle g_1, \ldots, g_s\rangle$ contains a monic polynomial. Since proving that $\mathrm{LT}(I)$ is finitely generated amounts to proving that $\mathrm{LT}(J)$ is finitely generated, one may suppose that I contains a monic polynomial. The desired result follows from Theorem 3 (a valuation domain obviously being coherent). □

Corollary 5 *For a semihereditary ring* $\mathbf{A}$*, the following assertions are equivalent:*

1. *For any finitely generated ideal* I *of* $\mathbf{A}[X]$*, the leading terms ideal* $\mathrm{LT}(I)$ *is also finitely generated.*
2. *If* J *is a finitely generated ideal of* $\mathbf{A}[X]$*, then* $J \cap \mathbf{A}$ *is finitely generated.*
3. $\dim \mathbf{A} \leq 1$.

Proof. The implications "*1.* ⇒ *2.* ⇒ *3.*" are as in [4].
"*3.* ⇒ *1.*" Follow the proof previously given for valuation domains applying the general dynamical technique as in [3, 4, 6, 7]. □

Final remark. Theorem 4 raises the following two questions:

Question 1: *Is it true that if* $\mathbf{V}$ *is a valuation ring with zero-divisors (i.e., a ring* $\mathbf{V}$ *such that for all* $a, b \in \mathbf{V}$*, either* a *divides* b *or* b *divides* a*) with Krull dimension* ≤ 1*, then for any finitely generated ideal* I *of* $\mathbf{V}[X]$*, the leading terms ideal of* I *is also finitely generated ?*

Question 2: *Is it true that if* $\mathbf{R}$ *is a domain with Krull dimension* ≤ 1*, then for any finitely generated ideal* I *of* $\mathbf{R}[X]$*, the leading terms ideal of* I *is also finitely generated ?*

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