Constructive aspects of Krull dimension

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Printable version of these slides

http://hlombardi.free.fr/publis/LectureDoc2.pdf

French detailed version

http://hlombardi.free.fr/publis/KroBasSer.pdf

Plan

- Introduction
- Constructive definitions.
- Krull boundaries.
- Elementary constructive definition of the Krull dimension.
- Kronecker theorem.
- Bass stable range theorem and Heitmann dimension.
- Other great theorems.

Introduction

Krull dimension was introduced by Krull in order to give an algebraic treatment of notions coming from analytic or differential geometry.

Basic properties of a good notion of dimension are (see Eisenbud):

- If $\mathbf{K}$ is a field, $\mathbf{K}[X_1, \ldots, X_n]$ and $\mathbf{K}[[X_1, \ldots, X_n]]$ must have dimension n.
- $\mathbb{C}\{\{X_1, \ldots, X_n\}\}$ must have dimension n.
- The notion has to be a "local" notion: the dimension of a global object is the maximum of its local dimensions.
- If $\mathbf{B} \supseteq \mathbf{A}$ is integral over $\mathbf{A}$ the two rings must have the same dimension.

Other desirable properties:

- Dimension of $\mathbf{A}[X]$ must be equal to $1+$ dimension of $\mathbf{A}$.
- Dimension has to decrease when passing to a quotient or a localization.

Introduction, 2.

At least in the Noetherian case, all desirable properties were proven to be true.

Nevertheless Krull dimension was defined in a very abstract way and did not seem to have a concrete content for the case of an arbitrary commutative ring.

Later, Krull dimension became a basic ingredient in many theorems of commutative algebra.

E.g., **Kronecker's theorem** says that *an algebraic variety in $\mathbb{C}^n$ can always be defined by $n+1$ equations*. This result has the following more general form:

In a commutative ring of dimension $\leq n$ every finitely generated ideal has the same radical as an ideal generated by $n+1$ elements.

This was noticed by van der Waerden in the Noetherian case, and much later extended by R. Heitmann to the non-Noetherian case.

Introduction, 3.

Krull dimension has also many consequences in the theory of **projective modules**.

A projective module is the algebraic analogue of a vector bundle in differential geometry. The analogue of a trivial vector bundle is a **free module**. Projective modules are locally free and an important question is to understand when a projective module is free.

A question of this kind was solved by Bass in the **Stable Range Theorem**: *if M is an $\mathbf{A}$-module such that $M \oplus \mathbf{A}^k \simeq \mathbf{A}^{n+k}$ and if the Krull dimension of $\mathbf{A}$ is $< n$ then M is free.*

This result was extended by R. Heitmann to the non-Noetherian case. Moreover a new abstract notion of dimension, more efficient for all questions concerning the projective modules has been introduced.

Introduction, 4.

In this lecture we show how various abstract notions of dimension have been recently made constructive and elementary, and how some important theorems related to these notions have become concrete theorems with an algorithmic content.

Some constructive definitions

- A ring $\mathbf{A}$ is **local** if: $x+y$ invertible implies x or y invertible.
- A ring $\mathbf{A}$ is **without zero divisor** if $xy = 0$ implies $x = 0$ or $y = 0$. A discrete domain is without zero divisor.
- A **filter** S in a ring $\mathbf{A}$ is the reciprocal image of the unit group $\mathbf{B}^\times$ by a ring homomorphism $\varphi : \mathbf{A} \to \mathbf{B}$.
  Equivalently: $1 \in S$ and $(xy \in S \iff x, y \in S)$.
- A **prime ideal** $\mathfrak{a} \subseteq \mathbf{A}$ is an ideal such that:
  $\mathbf{A}/\mathfrak{a}$ is without zero divisor and $(1 \in \mathfrak{a} \Rightarrow 1 = 0)$.
- A **prime filter** $S \subseteq \mathbf{A}$ is a filter such that:
  $S^{-1}\mathbf{A} = \mathbf{A}_S$ is a local ring and $(0 \in S \Rightarrow 1 = 0)$.
- A **saturated pair** $(\mathfrak{a}, S)$ in a ring $\mathbf{A}$ is the reciprocal image of $(0, \mathbf{B}^\times)$ by a ring homomorphism $\varphi : \mathbf{A} \to \mathbf{B}$. Equivalently:
  $\mathfrak{a}$ is an ideal, S is a filter, $\mathfrak{a}+S = S$ and $((sa \in \mathfrak{a},\ s \in S) \Rightarrow a \in \mathfrak{a})$.

Some constructive definitions, 2

- The **(nil)radical** of an ideal $\mathfrak{a}$ is the ideal

$$\mathrm{D}_{\mathbf{A}}(\mathfrak{a}) = \sqrt{\mathfrak{a}} = \{\, x \in \mathbf{A} \mid \exists n \in \mathbb{N}, x^n \in \mathfrak{a} \,\}.$$

  $\mathrm{N}(\mathbf{A}) = \mathrm{D}_{\mathbf{A}}(0)$ is the nilradical of $\mathbf{A}$.

  Notation: $\mathrm{D}_{\mathbf{A}}(x_1, \ldots, x_n) = \mathrm{D}_{\mathbf{A}}(\langle x_1, \ldots, x_n \rangle)$

- The **Jacobson radical** of an ideal $\mathfrak{a}$ is the ideal

$$\mathrm{J}_{\mathbf{A}}(\mathfrak{a}) = \{\, x \in \mathbf{A} \mid \forall y\, (1 + xy) \text{ is invertible modulo } \mathfrak{a} \,\}.$$

  $\mathrm{Rad}(\mathbf{A}) = \mathrm{J}_{\mathbf{A}}(0)$ is the (Jacobson) radical of $\mathbf{A}$.

  Notation: $\mathrm{J}_{\mathbf{A}}(x_1, \ldots, x_n) = \mathrm{J}_{\mathbf{A}}(\langle x_1, \ldots, x_n \rangle)$

- The **Zariski lattice of $\mathbf{A}$**, is the set
$$\mathsf{Zar}\,\mathbf{A} = \{\, \mathrm{D}_{\mathbf{A}}(x_1, \ldots, x_n) \,|\, n \in \mathbb{N},\, x_1, \ldots, x_n \in \mathbf{A} \,\}\,,$$
ordered by inclusion. This is a *distributive lattice* with
$$\mathrm{D}_{\mathbf{A}}(\mathfrak{a}_1) \vee \mathrm{D}_{\mathbf{A}}(\mathfrak{a}_2) = \mathrm{D}_{\mathbf{A}}(\mathfrak{a}_1 + \mathfrak{a}_2) \qquad \mathrm{D}_{\mathbf{A}}(\mathfrak{a}_1) \wedge \mathrm{D}_{\mathbf{A}}(\mathfrak{a}_2) = \mathrm{D}_{\mathbf{A}}(\mathfrak{a}_1\, \mathfrak{a}_2)$$
- The **Zariski spectrum of $\mathbf{A}$**, denoted by $\mathsf{Spec}\,\mathbf{A}$ is a (pointfree) topological spectral space whose compact opens form the lattice $\mathsf{Zar}\,\mathbf{A}$.
- In classical mathematics $\mathsf{Spec}\,\mathbf{A}$ has *enough points* (they are the prime ideals of $\mathbf{A}$), the compact open corresponding to $\mathrm{D}_{\mathbf{A}}(x_1, \ldots, x_n)$ is the set of prime ideals $\mathfrak{p}$ such that $x_1 \notin \mathfrak{p} \vee \cdots \vee x_n \notin \mathfrak{p}$.
- The **Heitmann lattice of $\mathbf{A}$**, is the set
$$\mathsf{Heit}\,\mathbf{A} = \{\, \mathrm{J}_{\mathbf{A}}(x_1, \ldots, x_n) \,|\, n \in \mathbb{N},\, x_1, \ldots, x_n \in \mathbf{A} \,\}\,,$$
ordered by inclusion, it is a *quotient distributive lattice* of $\mathsf{Zar}\,\mathbf{A}$.

Krull boundaries

Definition 1. *Let $\mathbf{A}$ be a commutative ring and $x \in \mathbf{A}$.*

1. *The* **boundary ideal of x in $\mathbf{A}$** *is*

$$\mathsf{K}_{\mathbf{A}}(x) = \mathsf{K}_{\mathbf{A}}^{x} := \langle x \rangle + (\mathsf{D}_{\mathbf{A}}(0) : x) \tag{1}$$

   The **Krull boundary quotient of x in $\mathbf{A}$** *is the quotient ring $\mathbf{A}/\mathsf{K}_{\mathbf{A}}(x)$.*

2. *The* **boundary monoid of x in $\mathbf{A}$** *is*

$$\mathsf{S}_{x}^{\mathsf{K}} = x^{\mathbb{N}}(1 + x\mathbf{A}) \tag{2}$$

   The **Krull boundary localization of x in $\mathbf{A}$** *is the localized ring $(\mathsf{S}_{x}^{\mathsf{K}})^{-1}\mathbf{A}$*

Elementary constructive definition of Krull dimension

A commutative ring $\mathbf{A}$ is said to have Krull dimension -1 when $\mathbf{A} = 0$. In classical mathematics this means that $\mathbf{A}$ has no prime ideal.

More generally, in classical mathematics a ring $\mathbf{A}$ is said to have Krull dimension $\leq d$ (and one writes $\mathsf{Kdim}\,\mathbf{A} \leq d$) if a chain of primes $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_{d+1}$ is impossible.

We have the following elementary characterization of this fact.

Theorem 2. *Let $\mathbf{A}$ be a commutative ring and an integer $\ell \geq 0$. T.F.A.E.*

1. $\mathsf{Kdim}\,\mathbf{A} \leq \ell$.
2. *For all $x \in \mathbf{A}$,* $\mathsf{Kdim}(\mathbf{A}/\mathsf{K}_{\mathbf{A}}(x)) \leq \ell - 1$.
3. *For all $x \in \mathbf{A}$,* $\mathsf{Kdim}((\mathsf{S}_x^{\mathsf{K}})^{-1}\mathbf{A}) \leq \ell - 1$.

Krull dimension, 2.

In particular $\mathsf{Kdim}\,\mathbf{A} \leq 0$ ($\mathbf{A}$ is zerodimensional) if and only if for all $x \in \mathbf{A}$ there exist $a \in \mathbf{A}$ and $n \in \mathbb{N}$ such that $x^n(1+ax) = 0$.

Similarly $\mathsf{Kdim}\,\mathbf{A} \leq 1$ if and only if for all $x, y \in \mathbf{A}$ there exist $a, b \in \mathbf{A}$ and $n, m \in \mathbb{N}$ such that $y^m(x^n(1+ax)+by) = 0$.

More generally Theorem 2 allows to give an inductive definition of the Krull dimension, manageable in constructive mathematics and equivalent to the classical one in classical mathematics.

A priori one can use an inductive definition based either on Krull boundary localizations or on Krull boundary quotients. One shows constructively that the two inductive definitions are equivalent and more precisely:

Proposition 3. *Let $\mathbf{A}$ be a commutative ring and an integer $\ell \geq 0$. T.F.A.E.*

(1) $\mathsf{Kdim}\,\mathbf{A} \leq \ell$

(2) *For all $x_0, \ldots, x_\ell \in \mathbf{A}$ there exist $b_0, \ldots, b_\ell \in \mathbf{A}$ such that*

$$\left.\begin{array}{rcl} \mathrm{D}_{\mathbf{A}}(b_0 x_0) & = & \mathrm{D}_{\mathbf{A}}(0) \\ \mathrm{D}_{\mathbf{A}}(b_1 x_1) & \leq & \mathrm{D}_{\mathbf{A}}(b_0, x_0) \\ \vdots & \vdots & \vdots \\ \mathrm{D}_{\mathbf{A}}(b_\ell x_\ell) & \leq & \mathrm{D}_{\mathbf{A}}(b_{\ell-1}, x_{\ell-1}) \\ \mathrm{D}_{\mathbf{A}}(1) & = & \mathrm{D}_{\mathbf{A}}(b_\ell, x_\ell) \end{array}\right\} \tag{3}$$

(3) *For all $x_0, \ldots, x_\ell \in \mathbf{A}$ there exist $a_0, \ldots, a_\ell \in \mathbf{A}$ and $m_0, \ldots, m_\ell \in \mathbb{N}$ such that*

$$x_0^{m_0}(x_1^{m_1} \cdots (x_\ell^{m_\ell}(1 + a_\ell x_\ell) + \cdots + a_1 x_1) + a_0 x_0) = 0 \tag{4}$$

Remark : The inequalities (3) in item (2) of proposition 3 give an interesting symmetric relation between the two sequences $(b_0, \ldots, b_\ell)$ and $(x_0, \ldots, x_\ell)$.

When $\ell = 0$, this means $\mathrm{D}_{\mathbf{A}}(b_0) \wedge \mathrm{D}_{\mathbf{A}}(x_0) = 0$ and $\mathrm{D}_{\mathbf{A}}(b_0) \vee \mathrm{D}_{\mathbf{A}}(x_0) = 1$, *i.e.*, the two elements $\mathrm{D}_{\mathbf{A}}(b_0)$ and $\mathrm{D}_{\mathbf{A}}(x_0)$ are complementary in the Zariski lattice $\mathsf{Zar}\,\mathbf{A}$. Inside $\mathsf{Spec}\,\mathbf{A}$ this means that the corresponding basic open sets are complementary.

We introduce the following terminology:
when two sequences $(b_0, \ldots, b_\ell)$ and $(x_0, \ldots, x_\ell)$ satisfy the inequalities (3) we say they are **complementary**. A sequence which has a complement is called a **singular sequence**.

Proposition 4.
Let $\mathbf{K}$ be a discrete field, $\mathbf{A}$ a $\mathbf{K}$-algebra, and $x_1, \ldots, x_\ell \in \mathbf{A}$ algebrically dependent over $\mathbf{K}$. The sequence $x_1, \ldots, x_\ell$ is singular.

Proof. Let $Q(x_1, \ldots, x_\ell) = 0$ be an algebraic dependence relation over $\mathbf{K}$. Using a lexicographic order on the monomials of Q we see it can be written as:

$$\begin{aligned} Q \;=\; & x_1^{m_1} \cdots x_\ell^{m_\ell} + x_1^{m_1} \cdots x_\ell^{1+m_\ell} R_\ell + x_1^{m_1} \cdots x_{\ell-1}^{1+m_{\ell-1}} R_{\ell-1} \\ & + \cdots + x_1^{m_1} x_2^{1+m_2} R_2 + x_1^{1+m_1} R_1 \end{aligned}$$

where $R_j \in \mathbf{K}[x_k\,;\, k \geq j]$. So we get an equality saying the sequence is singular.

Corollary 5.
If $\mathbf{K}$ is a discrete field, the Krull dimension of $\mathbf{K}[X_1, \ldots, X_\ell]$ is equal to ℓ.

Some other results easily obtained in a constructive way:

- If $\mathbf{B}$ is a localization or a quotient of $\mathbf{A}$, then $\mathsf{Kdim}\,\mathbf{B} \leq \mathsf{Kdim}\,\mathbf{A}$.
- If $(\mathfrak{a}_i)_{1\leq i\leq m}$ are ideals of $\mathbf{A}$ and $\mathfrak{a} = \bigcap_{i=1}^{m} \mathfrak{a}_i$, then $\mathsf{Kdim}(\mathbf{A}/\mathfrak{a}) = \sup_i \mathsf{Kdim}(\mathbf{A}/\mathfrak{a}_i)$.
- If $(S_i)_{1\leq i\leq m}$ are comaximal monoids of $\mathbf{A}$ then $\mathsf{Kdim}(\mathbf{A}) = \sup_i \mathsf{Kdim}(\mathbf{A}_{S_i})$.

More difficult:

- If $\mathbf{B} \supseteq \mathbf{A}$ is integral over $\mathbf{A}$ then $\mathsf{Kdim}\,\mathbf{B} = \mathsf{Kdim}\,\mathbf{A}$.

Kronecker theorem

Here is a simple but crucial lemma, reducing the number of "radically generators" of a radical ideal:

Lemma 6. *If* $u, v \in \mathbf{A}$ *one has*

$$\mathsf{D}_{\mathbf{A}}(u, v) \;=\; \mathsf{D}_{\mathbf{A}}(u+v, uv) \;=\; \mathsf{D}_{\mathbf{A}}(u+v) \vee \mathsf{D}_{\mathbf{A}}(uv)$$

In particular il $uv \in \mathsf{D}_{\mathbf{A}}(0)$, *then* $\mathsf{D}_{\mathbf{A}}(u, v) = \mathsf{D}_{\mathbf{A}}(u+v)$.

Proof. Obviously $\langle u+v, uv\rangle \subseteq \langle u, v\rangle$ so $\mathsf{D}_{\mathbf{A}}(u+v, uv) \subseteq \mathsf{D}_{\mathbf{A}}(u, v)$. On the other hand $u^2 = (u+v)u - uv \in \langle u+v, uv\rangle$ so $u \in \mathsf{D}_{\mathbf{A}}(u+v, uv)$. □

Kronecker theorem, 2.

Lemma 7. *Let $n \geq 1$. If $(b_1, \ldots, b_n)$ and $(x_1 \ldots, x_n)$ are two complementary sequences in $\mathbf{A}$ then for all $a \in \mathbf{A}$ one has:*

$$\mathrm{D}_{\mathbf{A}}(a, b_1, \ldots, b_n) = \mathrm{D}_{\mathbf{A}}(b_1 + ax_1, \ldots, b_n + ax_n),$$

So $a \in \mathrm{D}_{\mathbf{A}}(b_1 + ax_1, \ldots, b_n + ax_n)$.

Theorem 8.
(Kronecker Theorem, with Krull dimension, without Noetheriannity)
Let $n \geq 0$. If $\mathrm{Kdim}\,\mathbf{A} < n$ and $b_1, \ldots, b_n \in \mathbf{A}$ then there exist $x_1, \ldots, x_n$ such that for all $a \in \mathbf{A}$,

$$\mathrm{D}_{\mathbf{A}}(a, b_1, \ldots, b_n) = \mathrm{D}_{\mathbf{A}}(b_1 + ax_1, \ldots, b_n + ax_n).$$

Consequently, in a ring with Krull dimension $\leq n$, every finitely generated ideal has the same radical as an ideal generated by at most $n + 1$ elements.

Bass stable range theorem and Heitmann dimension

Theorem 9.
(Bass stable range, with Krull dimension, without Noetheriannity)
Let $n \geq 0$*. If* Kdim $\mathbf{A} < n$*, for all* $b_1, \dots, b_n \in \mathbf{A}$*, there exist* $x_1, \dots, x_n \in \mathbf{A}$ *such that*

$$\forall a \in \mathbf{A} \quad (1 \in \langle a, b_1, \dots, b_n \rangle \Rightarrow 1 \in \langle b_1 + ax_1, \dots, b_n + ax_n \rangle).$$

This theorem was first known in the Noetherian case. Then it was strengthened by replacing Krull dimension by the dimension of the maximal spectrum. Then Heitmann proved that there is a non-Noetherian version with a dimension (he calls this dimension Jdim) which generalizes the dimension of the maximal spectrum in the non-Noetherian case. This was defined in a rather complicated way, but finally it appeared later that the Jdim is the dimension of the spectral space corresponding to the Heitmann lattice.

Finally, trying to adapt the proof with Kdim for getting a proof with Jdim leads to a natural new inductive definition, which allows better proofs and improved theorems.

Definition 10.

1. *For* $x \in \mathbf{A}$ *the* **Heitmann boundary ideal of x in $\mathbf{A}$** *is:*

$$\mathsf{H}_{\mathbf{A}}(x) := \langle x \rangle + (\mathsf{J}_{\mathbf{A}}(0) : x)$$

2. *The* **Heitmann boundary quotient of x in $\mathbf{A}$** *is the quotient ring* $\mathbf{A}/\mathsf{H}_{\mathbf{A}}(x)$*.*

3. *The* **Heitmann dimension of $\mathbf{A}$** *is defined by induction:*
   (a) $\mathsf{Hdim}\,\mathbf{A} = -1$ *if and only if* $1_{\mathbf{A}} = 0_{\mathbf{A}}$*.*
   (b) *For* $\ell \geq 0$, $\mathsf{Hdim}\,\mathbf{A} \leq \ell$ *if and only if for all* $x \in \mathbf{A}$*,* $\mathsf{Hdim}(\mathbf{A}/\mathsf{H}_{\mathbf{A}}(x)) \leq \ell - 1$*.*

Since the Heitmann lattice is a quotient of the Krull lattice we get the following variant of the simple and crucial Lemma 6:

Lemma 11. *For $u, v \in \mathbf{A}$ we have*

$$\mathsf{J}_{\mathbf{A}}(u,v) \;=\; \mathsf{J}_{\mathbf{A}}(u+v, uv) \;=\; \mathsf{J}_{\mathbf{A}}(u+v) \vee \mathsf{J}_{\mathbf{A}}(uv)$$

In particular if $uv \in \mathsf{J}_{\mathbf{A}}(0)$, then $\mathsf{J}_{\mathbf{A}}(u,v) = \mathsf{J}_{\mathbf{A}}(u+v)$.

Then we deduce an improved Heitmann version of the Bass stable range.

Theorem 12. (Bass stable range, with Heitmann dimension) *Let $n \geq 0$. If $\mathsf{Hdim}\,\mathbf{A} < n$ and $1 \in \langle a, b_1, \ldots, b_n \rangle$ then there exist $x_1, \ldots, x_n$ such that $1 \in \langle b_1 + ax_1, \ldots, b_n + ax_n \rangle$.*

The preceding theorem is the key for the result concerning projective modules.

Corollary 13. *Let* $n \geq 0$*. If* $\mathsf{Hdim}\,\mathbf{A} < n$ *and* $V \in \mathbf{A}^{n+1}$ *is a unimodular vector, it can be transformed by elementary manipualtions in the basis vector* $(1, 0 \ldots, 0)$*.*

Corollary 14. *If* $\mathsf{Hdim}\,\mathbf{A} < n$ *an* $\mathbf{A}$*-module* M *such that* $M \oplus \mathbf{A}^k \simeq \mathbf{A}^{n+k}$ *is free.*

Serre splitting off and Foster Swan theorems

Theorem 15. (Serre's spillting off, with Heitmann dimension) *Let M be a projective $\mathbf{A}$-module of rank $\geq k$ over a ring $\mathbf{A}$ such that* $\mathsf{Hdim}\mathbf{A} < k$*. Then there exist an $\mathbf{A}$-module N and an isomorphism $M \simeq N \oplus \mathbf{A}$.*

Bass cancellation theorem