

# Comparison of Picard Groups in Dimension 1

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## Abstract

We compare two Picard groups in dimension one. Our proofs are constructive and the results generalize a theorem of J. Sands [11].

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## Introduction

In [11] J. Sands generalizes a theorem of Siegel [13]. He explains the link between the Picard group of the integer ring of a number field and that of an order of this number field. He applies his results to computing bounds on regulators.

We give here an extension of Sands' results in the case of two integral domains  $A \subseteq B$  of dimension 1 when  $B$  is finite over  $A$  with the same fraction field. Our proof is constructive and gives a possible algorithm for Computer Algebra (e.g., for computing the Picard group of algebraic curves) when hypotheses are satisfied in an explicit way.

Let us recall that the conductor  $c(A, B)$  of  $A$  in  $B$  (when  $A$  is a subring of  $B$ ) is an ideal of  $A$  and  $B$  defined by:

$$c(A, B) = \{x \in A \mid xB \subseteq A\}.$$

The fact that our theorem replaces the conductor of  $A$  in  $B$  by a nonzero ideal contained in the conductor, improves, w.r.t. the original theorem, the possibility of a concrete computation.

In the first section, we give partial results in the case of an arbitrary extension of rings. This gives as a particular case a construction of Schanuel related to seminormality. The second section is devoted to reminders concerning the constructive approach of Krull dimension. Section 3 gives the proof of the main theorem of this paper:

**Theorem 7** *Let  $A \subseteq B$  be two integral domains of dimension 1 with the same fraction field and such that  $B$  is a finitely generated  $A$ -module. If  $\mathfrak{f}$  is a nonzero ideal of  $B$  contained in the conductor of  $A$  in  $B$ , then we have the following exact sequence:*

$$1 \longrightarrow U(B)/U(A) \xrightarrow{i \ [b] \mapsto [b]} U(B/\mathfrak{f})/U(A/\mathfrak{f}) \xrightarrow{j \ [b] \mapsto [bA+\mathfrak{f}]} \text{Pic}(A) \xrightarrow{\pi \ [a] \mapsto [aB]} \text{Pic}(B) \longrightarrow 1$$

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Let us recall that this theorem was given by Dedekind [5] in algebraic number theory when  $B$  is the ring of all integers in a number field.

Our proofs are given in the usual style of constructive algebra (cf. [9]). We have no Noetherian hypothesis. We use a simple characterization “without prime ideals” of Krull dimension ([2]): this characterization has already lead to elementary and constructive proofs for some classical results in commutative algebra that were previously purely ideal theorems (cf. [3, 4, 6, 8]).

Our theorem is new from two points of view. First we don't use any Noetherian hypothesis, second we obtain an algorithm that makes explicit the exactness of the sequence.

Let us insist on the fact that theorems whose hypotheses mention the Krull dimension can have an algorithmic version only once Krull dimension has become an explicit notion.

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## 1 Exact sequences for groups of invertible modules

We note  $U(C)$  the group of invertible elements of an arbitrary ring  $C$ .

Let us consider two commutative rings  $A \subseteq B$ . We say that a sub- $A$ -module  $M$  of  $B$  is *invertible* if there exists a sub- $A$ -module  $N$  of  $B$  such that  $M.N = A$ . This definition coincides with that of Bourbaki [1] when  $B$  is the localization of  $A$  at a multiplicative subset made up of non-zerodivisors. In this case we get for any sub- $A$ -module  $M'$  of  $B$  that  $M.M' \simeq M \otimes_A M'$  via the canonical homomorphism. Indeed, we have  $x_1, \dots, x_n$  in  $M$ ,  $y_1, \dots, y_n$  in  $N$  such that  $1 = \sum_i x_i y_i$  and  $x_i y_j$  in  $A$ . For any element  $\sum_k z_k \otimes z'_k$  in  $M \otimes_A M'$ , since  $y_i z_k \in N.M = A$ , we have

$$\sum_k z_k \otimes z'_k = \sum_{k,i} x_i y_i z_k \otimes z'_k = \sum_{k,i} x_i (y_i z_k) \otimes z'_k = \sum_{k,i} x_i \otimes (y_i z_k) z'_k = \sum_i x_i \otimes \sum_k y_i z_k z'_k,$$

so the canonical surjection  $M \otimes_A M' \rightarrow M.M'$  is injective. Thus invertible modules are rank 1 projective.

We note  $\mathcal{M}_{\text{inv}}(A, B)$  the group of invertible sub- $A$ -modules of  $B$  and  $\mathcal{I}_{\text{fr}}(B)$  the group of invertible fractionary ideals of  $B$ : in other words  $\mathcal{I}_{\text{fr}}(B) = \mathcal{M}_{\text{inv}}(B, \text{Frac}(B))$ , where  $\text{Frac}(B)$  is the total fraction ring of  $B$ . We note  $\mathcal{P}_{\text{fr}}(A, B)$  the sub-group of  $\mathcal{M}_{\text{inv}}(A, B)$  made up of monogeneous sub- $A$ -modules of  $B$ . Finally  $\text{Pic}(A)$  is the group isomorphism classes of rank 1 projective modules.

### 1.1 First exact sequence

Let us consider an ideal  $\mathfrak{f}$  of  $B$  such that  $\mathfrak{f}B \subseteq A$ , i.e.,  $\mathfrak{f} \subseteq \mathfrak{c}(A, B)$ . We define a first exact sequence

$$1 \longrightarrow U(A/\mathfrak{f}) \longrightarrow U(B/\mathfrak{f}) \xrightarrow{j} \mathcal{M}_{\text{inv}}(A, B)$$

- Since  $\mathfrak{f}$  is an ideal of  $A$  and  $B$ , one has  $A/\mathfrak{f} \subseteq B/\mathfrak{f}$ . So  $U(A/\mathfrak{f}) \subseteq U(B/\mathfrak{f})$ . This defines the (injective) left arrow.

- Let us look at the arrow  $j$ . For  $b \in B$ , let  $j(b) = bA + \mathfrak{f}$ . Up to now  $j(b)$  is only a sub- $A$ -module of  $B$ . Clearly  $j(1) = A$ , and  $b_1 \equiv b_2 \pmod{\mathfrak{f}}$  implies  $j(b_1) = j(b_2)$ . Remark that if  $b_1 \in B$  is invertible modulo  $\mathfrak{f}$ , then  $b_1 \mathfrak{f} + \mathfrak{f}^2 = \mathfrak{f}$  (multiplying  $b_1 B + \mathfrak{f} = B$  by  $\mathfrak{f}$ ). For all  $b_2 \in B$ , we get  $j(b_1 b_2) = j(b_1) j(b_2)$  since

$$j(b_1) j(b_2) = b_1 b_2 A + b_1 \mathfrak{f} + b_2 \mathfrak{f} + \mathfrak{f}^2 = b_1 b_2 A + (b_1 B + \mathfrak{f}) \mathfrak{f} + b_2 \mathfrak{f} = b_1 b_2 A + \mathfrak{f} = j(b_1 b_2).$$

This allows us to define the arrow  $j : U(B/\mathfrak{f}) \mapsto \mathcal{M}_{\text{inv}}(A, B)$  that sends the class modulo  $\mathfrak{f}$  of an (invertible modulo  $\mathfrak{f}$ ) element  $b \in B$  to  $bA + \mathfrak{f}$  (this is an invertible  $A$ -module because  $j$  is multiplicative).

Remark that  $j(b)B = B$  for all  $b \in U(B/\mathfrak{f})$ .

In order to show the exactness of the desired sequence, we have to compute  $\ker j$ . Let  $b$  be an element in  $B$  invertible modulo  $\mathfrak{f}$ , such that  $j(b) = A$ , i.e.,  $\mathfrak{f} + bA = A$ . Then  $bA \subseteq A$ ,  $b \in A$  and the equality  $\mathfrak{f} + bA = A$  says that the element  $b \in A$  is invertible modulo  $\mathfrak{f}$ .

In the sequel, when  $b \in B$  we use without more precision the notations  $\bar{b}$ ,  $\widehat{b}$ ,  $\widetilde{b}$  for the image of  $b$  in a set  $X$  through some natural map  $B \rightarrow X$ .

The previous result can be rephrased in the following way:

**Proposition 1** *The map  $j : U(B/\mathfrak{f}) \rightarrow \mathcal{M}_{\text{inv}}(A, B)$  defined by  $j(\bar{b}) = bA + \mathfrak{f}$  induces an isomorphism from  $U(B/\mathfrak{f})/U(A/\mathfrak{f})$  onto some sub-group of  $\mathcal{M}_{\text{inv}}(A, B)$  made up of invertible  $A$ -modules  $\mathfrak{a}$  of  $B$  such that  $\mathfrak{a}B = B$ .*

## 1.2 Second exact sequence

We assume in the sequel that  $B$  is integral over  $A$ .

The kernel of the homomorphism  $U(B) \rightarrow U(B/\mathfrak{f})/U(A/\mathfrak{f})$  is equal to  $U(A)$  (an element of  $A$  which is invertible in  $B$  is invertible in  $A$ ). This gives the arrow  $i$  in “the second exact sequence”:

$$1 \longrightarrow U(B)/U(A) \xrightarrow{i} U(B/\mathfrak{f})/U(A/\mathfrak{f}) \xrightarrow{j} \mathcal{M}_{\text{inv}}(A, B)/\mathcal{P}_{\text{fr}}(A, B)$$

Concerning the exactness it remains to compute  $\ker j$  (we keep the same name  $j$ ). Let  $b \in B$  be a unit modulo  $\mathfrak{f}$  such that  $j(b)$  is principal, i.e.,  $bA + \mathfrak{f} = b'A$  with  $b' \in B$ . Multiplying this equality by  $B$ , one obtains  $bB + \mathfrak{f} = b'B$ , i.e.,  $B = b'B$ , and so  $b'$  is invertible in  $B$ . Multiplying  $bA + \mathfrak{f} = b'A$  by  $b'^{-1}$  and letting  $a = bb'^{-1}$ , one gets  $aA + \mathfrak{f} = A$  so that  $a \in A$  and is invertible modulo  $\mathfrak{f}$ . Finally  $\bar{b} = i(\widehat{b'})$  as claimed.

The exactness of the above sequence may be rephrased in the following way.

**Proposition 2** *We assume that  $B$  is integral over  $A$ . The map  $j : U(B/\mathfrak{f})/U(A/\mathfrak{f}) \rightarrow \mathcal{M}_{\text{inv}}(A, B)$  defined by  $j(\bar{b}) = bA + \mathfrak{f}$  induces an isomorphism of  $U(B/\mathfrak{f})/U(A/\mathfrak{f})U(B)$  onto a sub-group of  $\mathcal{M}_{\text{inv}}(A, B)/\mathcal{P}_{\text{fr}}(A, B)$ . In other words, for an element  $b \in B$  which is invertible modulo  $\mathfrak{f}$ , the sub- $A$ -module  $bA + \mathfrak{f}$  is principal if and only if there exists  $u \in U(B)$  such that  $ub \in A$ .*

Let us precise “in other words”: let  $b \in B$  be an element which is invertible modulo  $\mathfrak{f}$ . Its class is in  $U(A/\mathfrak{f})U(B)$  if and only if  $b = ab' \pmod{\mathfrak{f}}$  with  $a \in A$  invertible modulo  $\mathfrak{f}$  and  $b' \in U(B)$ . In this case we take  $u = b'^{-1}$ . Conversely, if such an  $u$  exists, the element  $a = ub$  is in  $A$  and its class modulo  $\mathfrak{f}$  is invertible in  $B/\mathfrak{f}$ , and thus invertible in  $A/\mathfrak{f}$  since  $B/\mathfrak{f}$  is integral over  $A/\mathfrak{f}$ .

## 1.3 Application: seminormality

Let  $A \subseteq A[\alpha]$  with  $\alpha^2$  and  $\alpha^3 \in A$ . Let  $B = A[\alpha] = A + \alpha A$  and  $\mathfrak{f} = \alpha^2 B = \alpha^2 A + \alpha^3 A$ . For any  $a \in A$  the element  $b = 1 + a\alpha$  is invertible modulo  $\mathfrak{f}$  (its inverse is  $1 - \alpha a$ ) and the sub- $A$ -module  $j(\bar{b})$  is invertible. By definition  $j(\bar{b}) = bA + \alpha^2 A + \alpha^3 A$  is also equal to  $bA + \alpha^2 A$  or  $bA + \alpha^3 A$  (since  $\alpha^3 = \alpha^3 b - (a\alpha^2)\alpha^2$  and  $\alpha^2 = \alpha^2 b - a\alpha^3$ ).

Consequently we get a homomorphism  $\varphi : (A, +) \rightarrow \text{Pic}(A)$  by composing  $A \rightarrow U(B/\mathfrak{f})$ ,  $a \mapsto \overline{1 + a\alpha}$  and  $j$ .

Now assume that  $A[\alpha]$  is reduced and replace  $A$  by  $A[X]$ ,  $B$  by  $B[X]$  and  $\alpha$  by  $\alpha X$ . We get a homomorphism  $\varphi : (A[X], +) \rightarrow \text{Pic}(A[X])$  which maps  $P \in A[X]$  to the class of  $M(X) = (1 + \alpha X P)A[X] + (\alpha X)^2 A[X]$  (an  $A[X]$ -projective module). According to Proposition 2, this module is free if and only if there exists  $V \in U(B[X])$  such that  $V(1 + \alpha X P) \in A[X]$ . Since  $B$  is reduced we

have  $U(B[X]) = U(B)$ . This gives the condition  $\exists v \in U(B), v + \alpha vXP \in A[X]$ , i.e.,  $v \in A$  and  $\alpha vP \in A[X]$ . Since  $v$  is invertible in  $B$  it is invertible in  $A$  and the module  $M(X)$  is free if and only if  $\alpha P \in A[X]$ . In case  $P = 1$  we retrieve ‘‘Schanuel’s example’’, namely  $M(0)$  is free, and  $M(X)$  is extended (i.e., free) if and only if  $\alpha \in A$ . Concerning this topic see [7] pages 29-30 and 39-40.

## 2 Reminder about dimension 1 (in constructive mathematics)

One finds a constructive definition of Krull dimension in [2]. Applications of this notion have been given in [3, 4, 6, 8].

The trivial ring is characterized by dimension  $-1$ . Concerning dimensions 0 and 1, one has the following elementary characterizations.

**Lemma 3** (dimensions 0 and 1) *Let  $A$  be a commutative ring.*

1. *The ring  $A$  is of dimension  $\leq 0$  if and only if for all  $x \in A$  there exist  $n \in \mathbb{N}$  and  $a \in A$  such that  $x^n(1 + ax) = 0$ .*
2. *The ring  $A$  is of dimension  $\leq 1$  if and only if for all  $x, y \in A$  there exist  $m, n \in \mathbb{N}$  and  $a, b \in A$  such that  $y^m(x^n(1 + ax) + by) = 0$ .*

Moreover ‘‘usual rings’’ which are of dimension  $\leq 0$  or 1 in classical mathematics satisfy in an explicit way the above elementary characterization.

The following lemma appears with a constructive proof in a slightly more general form in [6] (Lemma 4.3).

**Lemma 4** (avoiding lemma in dimension 1) *Let  $A$  be an integral domain of dimension  $\leq 1$ . Let  $\mathfrak{a}$  be an invertible ideal of  $A$  and  $\mathfrak{b}$  a nonzero ideal. Then there exists an element  $u \neq 0$  of  $\text{Frac}(A)$  such that the ideal  $u\mathfrak{a}$  is integral (i.e., contained in  $A$ ) and comaximal to  $\mathfrak{b}$ .*

The famous ‘‘one and a half theorem’’ is also constructively proven in [6] (Theorem 2.32) in the following general form.

**Theorem 5** (one and a half theorem) *Let  $A$  be a commutative ring of dimension  $\leq 1$  and  $\mathfrak{a}$  an invertible ideal. Let  $x \in \mathfrak{a}$  be a non-zerodivisor. Then there exists  $y \in \mathfrak{a}$  such that for all  $n \geq 1$ ,  $\mathfrak{a} = x^n\mathfrak{a} + yA$ . In particular,  $\mathfrak{a} = \langle x^n, y \rangle$ .*

## 3 An exact sequence for class groups

### 3.1 Context

Let us consider two integral domains  $A \subseteq B$  having the same fraction field.

The conductor  $c(A, B)$  of  $A$  in  $B$  is the annihilator of the  $A$ -module  $B/A$ , and this is also the greatest ideal of  $B$  contained in  $A$ .

We assume moreover that  $A$  is a ring of dimension 1 and  $B$  is a finitely generated  $A$ -module. So  $B$  has dimension 1.

The hypotheses ‘‘ $\text{Frac}(A) = \text{Frac}(B)$ ’’ and ‘‘ $B$  is a finitely generated  $A$ -module’’ imply that  $B$  is a fractionary ideal of  $A$  (consider a finite system of generators of  $B$  over  $A$  and multiply by a common  $A$ -denominator). In fact,  $\text{Frac}(A) = \text{Frac}(B)$  says that  $B/A$  is a torsion  $A$ -module and the fact that  $B$  is a finitely generated  $A$ -module (equivalently,  $B/A$  is a finitely generated  $A$ -module) implies that its annihilator (i.e., the conductor of  $A$  in  $B$ ) is not reduced to 0.

This context appears in algebraic number theory, namely  $B$  is the ring of all integral numbers of a number field  $K$  and  $A$  is a number ring in  $K$  (i.e., a subring of  $B$  which is a  $\mathbb{Z}$ -module of rank equal to the dimension of  $K$  as a  $\mathbb{Q}$ -vector space).

There are other applications of this exact sequence. For example in geometry where rings  $A$  and  $B$  are finitely presented  $\mathbf{k}[T]$ -algebras (coordinate rings of affine curves). See [10] and the comment of J.-P. Serre in [12].

In the sequel we consider a nonzero ideal  $\mathfrak{f}$  of  $B$  contained in the conductor  $c(A, B)$ .

### 3.2 Back to the first exact sequence

We get the following analogue of the first exact sequence (section 1.1):

$$1 \longrightarrow U(A/\mathfrak{f}) \longrightarrow U(B/\mathfrak{f}) \xrightarrow{j} \mathcal{I}_{\text{fr}}(A) \xrightarrow{\pi} \mathcal{I}_{\text{fr}}(B)$$

where  $\pi$  is the canonical map  $\mathfrak{a} \mapsto \mathfrak{a}B$  and  $j$  is again defined by  $\bar{b} \mapsto bA + \mathfrak{f}$ . Proofs of the exactness up to  $\mathcal{I}_{\text{fr}}(A)$  are identical to that of section 1.1 and it remains to compute  $\ker \pi$ . Let  $\mathfrak{a}$  be a fractionary ideal of  $A$  such that  $\mathfrak{a}B = B$ . Multiplying this equality by  $\mathfrak{f}$ , we get  $\mathfrak{a}\mathfrak{f} = \mathfrak{f}$  and so  $\mathfrak{f} \subseteq \mathfrak{a}$ . By virtue of Theorem 5, there exists  $b \in \mathfrak{a}$  such that  $bA + \mathfrak{f} = \mathfrak{a}$ . We multiply this equality by  $B$ , we get  $bB + \mathfrak{f} = \mathfrak{a}B = B$ ,  $b \in B$  and  $b$  is invertible modulo  $\mathfrak{f}$ , that is,  $\mathfrak{a} = j(\bar{b})$ .

This gives the following variant of Proposition 1.

**Proposition 6** *The map  $j : U(B/\mathfrak{f}) \rightarrow \mathcal{I}_{\text{fr}}(A)$  defined by  $j(\bar{b}) = bA + \mathfrak{f}$  induces an isomorphism from  $U(B/\mathfrak{f})/U(A/\mathfrak{f})$  onto the sub-group of  $\mathcal{I}_{\text{fr}}(A)$  made up of invertible fractionary ideals  $\mathfrak{a}$  of  $A$  such that  $\mathfrak{a}B = B$ .*

### 3.3 The canonical exact sequence

The canonical exact sequence is the following analogue of the second exact sequence (section 1.2):

$$1 \longrightarrow U(B)/U(A) \longrightarrow U(B/\mathfrak{f})/U(A/\mathfrak{f}) \xrightarrow{j} \text{Pic}(A) \xrightarrow{\pi} \text{Pic}(B) \longrightarrow 1$$

Since  $B$  is an integral domain, the group  $\text{Pic}(B)$  is isomorphic to  $\mathcal{I}_{\text{fr}}(B)/\mathcal{P}_{\text{fr}}(B)$ . If  $B$  is a Prüfer domain  $\text{Pic}(B) = \mathcal{I}_{\text{fr}}(B)/\mathcal{P}_{\text{fr}}(B)$  is the usual class group of finitely generated (fractionary) ideals.

**Theorem 7** *Let  $A \subseteq B$  be two integral domains of dimension 1 with the same fraction field and such that  $B$  is a finitely generated  $A$ -module. If  $\mathfrak{f}$  is a nonzero ideal of  $B$  contained in the conductor of  $A$  in  $B$ , then we have the following exact sequence:*

$$1 \longrightarrow U(B)/U(A) \xrightarrow{i \ [b] \mapsto [b]} U(B/\mathfrak{f})/U(A/\mathfrak{f}) \xrightarrow{j \ [b] \mapsto [bA + \mathfrak{f}]} \text{Pic}(A) \xrightarrow{\pi \ [a] \mapsto [aB]} \text{Pic}(B) \longrightarrow 1$$

**Proof.**

It remains to prove that  $\pi$  is onto. Let  $\mathfrak{b}$  be an invertible fractionary ideal of  $B$ . Applying the avoiding lemma in dimension 1 (Lemma 4), we can assume that  $\mathfrak{b}$  is an integral ideal of  $B$  comaximal to  $\mathfrak{f}$ . Let  $1 = b + f$  with  $b \in \mathfrak{b}$ ,  $f \in \mathfrak{f}$  and let us denote  $\mathfrak{b}' = \mathfrak{b}\mathfrak{b}^{-1}$ . We have  $b \in A$  because  $\mathfrak{f} \subseteq A$ . Moreover  $\mathfrak{b}'$  is an integral ideal of  $B$  comaximal to  $\mathfrak{f}$  (since  $1 = b + f \in \mathfrak{b}' + \mathfrak{f}$ ) satisfying  $\mathfrak{b}\mathfrak{b}' = bB$ . Let  $\mathfrak{a} = \mathfrak{b} \cap A$  and  $\mathfrak{a}' = \mathfrak{b}' \cap A$ , Lemma 8 below implies that  $\mathfrak{a}B = \mathfrak{b}$  and  $\mathfrak{a}\mathfrak{a}' = bA$ . So  $\mathfrak{a}$  is invertible in  $A$  and  $\pi([\mathfrak{a}]) = [\mathfrak{b}]$ .  $\square$

**Lemma 8** (avoiding conductor lemma, Dedekind, cf. Sands [11] Theorem 3.1)

*Let  $A \subseteq B$  be two rings. Let  $\mathfrak{f}$  be an ideal of  $B$  contained in  $c(A, B)$ . Then the two maps  $\mathfrak{a} \mapsto \mathfrak{a}B$  and  $\mathfrak{b} \mapsto \mathfrak{b} \cap A$  are reciprocal bijections one of the other. Moreover, they preserve intersection, sum and product when restricted to integral ideals comaximal to  $\mathfrak{f}$ .*

**Proof.**

We only show the first item. Let  $\mathfrak{a}$  be an ideal of  $A$  such that  $\mathfrak{a} + \mathfrak{f} = A$ . Then  $\mathfrak{a}B \cap A = \mathfrak{a}$ . Indeed, we have  $a + f = 1$  with  $a \in \mathfrak{a}$  and  $f \in \mathfrak{f}$ . If  $a' \in \mathfrak{a}B \cap A$ , then  $a' = aa' + a'f$  with  $aa' \in \mathfrak{a}$  and  $a'f \in \mathfrak{a}Bf \subseteq \mathfrak{a}A = \mathfrak{a}$ .

Let  $\mathfrak{b}$  be an ideal of  $B$  such that  $\mathfrak{b} + \mathfrak{f} = B$  and let us show that  $(\mathfrak{b} \cap A)B = \mathfrak{b}$ . We have  $b + f = 1$  with  $b \in \mathfrak{b}$  and  $f \in \mathfrak{f}$ , thus  $b \in \mathfrak{b} \cap A$ . If  $b' \in \mathfrak{b}$ , then  $b' = bb' + b'f$  with  $bb' \in (\mathfrak{b} \cap A)B$  and  $b'f \in \mathfrak{b} \cap A$ .  $\square$

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