Smooth parametrizations for several cases of the Positivstellensatz

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Abstract

We give smooth parametrizations ($\mathcal{C}^r$, $\mathcal{C}^\infty$ or Nash) for several cases of the Positivstellensatz. This improves previous results where the parametrizations were obtained by means of semipolynomials (sup–inf combination of polynomial functions). A general homogeneous Positivstellensatz is obtained and used to get several homogeneous versions of the parametrization theorems.

Introduction.

This paper is a variation on the subject considered in [DGL] with several improvements. These improvements are shown with three examples. Let $\mathbf{K}$ be an ordered field and $\mathbf{R}$ its real closure.

Our first example is obtained by considering Hilbert's 17^{th} Problem. Let $\mathbf{x}$ denote the n variables $(x_1,\ldots,x_n)$. If $f(\mathbf{x}) = f(x_1,\ldots,x_n)$ is a polynomial in $\mathbf{K}[\mathbf{x}] = \mathbf{K}[x_1,\ldots,x_n]$ everywhere nonnegative in $\mathbf{R}^n$ then it can be considered as the specialization of the general polynomial $f_{n,d}(\mathbf{c},\mathbf{x})$ of degree d in $\mathbf{x}$ with coefficients $\mathbf{c} = (c_1,\ldots,c_m)$ where $m = \binom{n+d}{n}$. This specialization will be obtained by replacing $\mathbf{c}$ by a point of the closed $\mathbb{Q}$-semialgebraic set defined by:

$$\mathbb{F}_{n,d}(\mathbf{R}) = \{\gamma \in \mathbf{R}^m : \forall \xi \in \mathbf{R}^n \quad f_{n,d}(\gamma,\xi) \geq 0\}.$$

A $\mathbf{K}$-semipolynomial (defined, e.g., in [DGL] or [GL_1]), also called a sup–inf–polynomially $\mathbf{K}$–definable (SIPD) function, is a suprema of infima of finitely many polynomials in $\mathbf{K}[y_1,\ldots,y_t]$.

Theorem A (Rational $\mathcal{C}^r$ parametrization of Hilbert's 17^{th} Problem)
There exists a linear form $h_{n,d}(\mathbf{c})$ with integer coefficients such that:

$$\gamma \in \mathbb{F}_{n,d}(\mathbf{R}) \setminus (0,\ldots,0) \quad \Longrightarrow \quad h_{n,d}(\gamma) > 0.$$

For every positive integer r, the polynomial $h_{n,d}(\mathbf{c})f_{n,d}(\mathbf{c},\mathbf{x})$ can be written as a sum of rational functions

$$h_{n,d}(\mathbf{c})f_{n,d}(\mathbf{c},\mathbf{x}) = \sum_j p_j(\mathbf{c})\left(\frac{q_j(\mathbf{c},\mathbf{x})}{k(\mathbf{c},\mathbf{x})}\right)^2 \qquad (\circ)$$

where

- *$k(\mathbf{c},\mathbf{x})$ and the $q_j(\mathbf{c},\mathbf{x})$ are polynomials in the variables $\mathbf{x}$ whose coefficients are homogeneous $\mathbb{Q}$-semipolynomials of class $\mathcal{C}^r$,*
- *the $p_j(\mathbf{c})$ are homogeneous $\mathbb{Q}$-semipolynomials of class $\mathcal{C}^r$.*
- *If $\gamma \in \mathbb{F}_{n,d}(\mathbf{R})$ then $k(\gamma,\mathbf{x})$ vanishes only on the zeros of $f_{n,d}(\gamma,\mathbf{x})$ and the nonnegativity of $p_j(\gamma)$ is "clearly" evident from its construction.*
- *Every summand in the right hand side of $(\circ)$ is $\mathbf{c}$–homogeneous with $\mathbf{c}$–degree equal to 2.*

Remark that, as in [DGL], every summand in $(\circ)$ is a well defined continuous rational function when $\gamma \in \mathbb{F}_{n,d}(\mathbf{R})$ but otherwise it is only possible to guarantee the equality in $(\circ)$ when mutlipied by $k(\mathbf{c},\mathbf{x})^2$.

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Next we consider a variation of Hilbert's 17^{th} Problem concerning homogeneous polynomials everywhere positive. If $g(\mathbf{x})$ is a homogeneous polynomial with even degree then it can be regarded as the specialization of the general $\mathbf{x}$-homogeneous polynomial $g_{n,d}(\mathbf{c},\mathbf{x})$ of degree d in $\mathbf{x}$ with coefficients $\mathbf{c}=(c_1,\ldots,c_{m'})$ where $m'=\binom{n+d-1}{n-1}$. Consider the $\mathbb{Q}$-semialgebraic set $\mathbb{U}_{n,d}$ defined by:

$$\mathbb{U}_{n,d}(\mathbf{R})=\{\gamma\in\mathbf{R}^{m'}:\forall\xi\in\mathbf{R}^n\setminus\{(0,\ldots,0)\}\quad g_{n,d}(\gamma,\xi)>0\}.$$

Since $\mathbb{U}_{n,d}(\mathbf{R})$ is the set of γ's such that $g_{n,d}(\gamma,\xi)>0$ for any ξ in the unit sphere, and since this sphere is compact (in the semialgebraic sense), the set $\mathbb{U}_{n,d}(\mathbf{R})$ is clearly open.

Theorem B (Parametrizations for a variant of the Homogeneous Hilbert's 17^{th} Problem)
Let c_1 be the coefficient of x_1^d in $g_{n,d}(\mathbf{c},\mathbf{x})$ (which is positive if $\mathbf{c}=\gamma\in\mathbb{U}_{n,d}(\mathbf{R})$). The polynomial $c_1g_{n,d}(\mathbf{c},\mathbf{x})$ (d even) can be written as a sum of rational functions

$$c_1g_{n,d}(\mathbf{c},\mathbf{x})=p_1(\mathbf{c})\left(\frac{\|\mathbf{x}\|^s}{k(\mathbf{c},\mathbf{x})}\right)^2+\sum_{j\geq 2}p_j(\mathbf{c})\left(\frac{q_j(\mathbf{c},\mathbf{x})}{k(\mathbf{c},\mathbf{x})}\right)^2 \qquad (\circ\circ)$$

with $s>0$,

- *$k(\mathbf{c},\mathbf{x})$ and the $q_j(\mathbf{c},\mathbf{x})$ are homogeneous polynomials in the variables $\mathbf{x}$, the $\mathbf{x}$-degree of every rational function in the sum being equal to d and*
- *if $\gamma\in\mathbb{U}_{n,d}(\mathbf{R})$ then $p_1(\gamma)$ is strictly positive, the $p_j(\gamma)$ ($j\geq 2$) are nonnegative and $k(\gamma,\xi)$ is different from 0 when $\xi\neq(0,\ldots,0)$.*

Concerning the coefficients in the right hand side of the previous equality, three different types of parametrization are obtained:

- $\star$ *For any fixed integer $r\geq 0$, the $p_j(\mathbf{c})$ and the $\mathbf{x}$-coefficients of $k(\mathbf{c},\mathbf{x})$ and the $q_j(\mathbf{c},\mathbf{x})$ are $\mathbb{Q}$-semipolynomials homogeneous of class $\mathcal{C}^r$, the $\mathbf{c}$-degrees of the summands in the considered equality are equal to 2 and the positivity of $p_1(\gamma)$ and the nonnegativity of $p_j(\gamma)$ ($j\geq 2$), when $\gamma\in\mathbb{U}_{n,d}(\mathbf{R})$, are "clearly" evident from their construction.*
- $*$ *The $p_j(\mathbf{c})$ and the $\mathbf{x}$-coefficients of $k(\mathbf{c},\mathbf{x})$ and the $q_j(\mathbf{c},\mathbf{x})$ are semialgebraic and continuous throughout $\mathbf{R}^{m'}$, Nash on $\mathbb{U}_{n,d}(\mathbf{R})$ and vanishing outside $\mathbb{U}_{n,d}(\mathbf{R})$.*
- $\diamond$ *When $\mathbf{R}=\mathbb{R}$, the $p_j(\mathbf{c})$ and the $\mathbf{x}$-coefficients of $k(\mathbf{c},\mathbf{x})$ and the $q_j(\mathbf{c},\mathbf{x})$ are $\mathcal{C}^\infty$ throughout $\mathbb{R}^{m'}$, analytic on $\mathbb{U}_{n,d}(\mathbb{R})$ and vanishing outside $\mathbb{U}_{n,d}(\mathbb{R})$.*

The same remark made after Theorem A concerning the continuity of the summands in ($\circ$) applies to the summands in ($\circ\circ$).

Finally we present a parametrized homogeneous real Nullstellensatz which can be considered as a variation of Theorem B. This is obtained by considering a list of p general homogeneous polynomials with n variables (everyone with weight equal to 1) and fixed degrees. Let $\mathrm{lst}=(n,d_1,\ldots,d_p)$ be a list of positive integers, m_i be equal to $\binom{n+d_i-1}{n-1}$ and

$$m''=m_1+\cdots+m_p.$$

We consider $g_i(\mathbf{c}_i,\mathbf{x})$ the general homogeneous polynomial of $\mathbf{x}$-degree d_i with coefficients $\mathbf{c}_i=(c_1^{(i)},\ldots,c_{m_i}^{(i)})$. Finally let $\mathbf{c}$ be equal to $(\mathbf{c}_1,\ldots,\mathbf{c}_p)$. We define $W_{\mathrm{lst}}(\mathbf{R})$ as the set of all the $\gamma=(\gamma_1,\ldots,\gamma_p)\in\mathbf{R}^{m''}$ such that the system of polynomial equations

$$g_1(\gamma_1,\mathbf{x})=0,\ \ldots,\ g_p(\gamma_p,\mathbf{x})=0$$

has no solutions except $(0,\ldots,0)$. $W_{\mathrm{lst}}(\mathbf{R})$ is a $\mathbb{Q}$-semialgebraic set which is open by a compactness argument since

$$\gamma\in W_{\mathrm{lst}}(\mathbf{R})\quad\Longleftrightarrow\quad\forall\xi\in S_n(\mathbf{R})\quad\sum_{i=1}^{p}(g_i(\gamma,\xi))^2>0,$$

where $S_n(\mathbf{R})$ is the unit sphere in $\mathbf{R}^n$.

Theorem C (Parametrization of the weak real homogeneous Nullstellensatz)
For every list $\mathrm{lst} = (n, d_1, \ldots, d_p)$*, we obtain an algebraic identity:*

$$p_1(\mathbf{c})\|\mathbf{x}\|^{2s} + \sum_{j\geq 2} p_j(\mathbf{c})a_j(\mathbf{c},\mathbf{x})^2 + \sum_{i=1}^{p} g_i(\mathbf{c},\mathbf{x})b_i(\mathbf{c},\mathbf{x})^2 = 0$$

with $s > 0$*,*

- *the* $a_j(\mathbf{c},\mathbf{x})$ *and* $b_j(\mathbf{c},\mathbf{x})$ *are* $\mathbf{x}$*-homogeneous polynomials, the* $\mathbf{x}$*-degree of every term in the previous equation being equal to* $2s$*, and*
- *if* $\gamma \in W_{\mathrm{lst}}(\mathbf{R})$*, then* $p_1(\gamma) > 0$ *and, for* $j \geq 2$*,* $p_j(\gamma) \geq 0$*.*

Concerning the $\mathbf{x}$*-coefficients in the left hand side of the previous equation we obtain three types of parametrization:*

⋆ *For any fixed integer* $r \geq 0$*, the* $p_j(\mathbf{c})$ *and the* $\mathbf{x}$*-coefficients of the* $a_j(\mathbf{c},\mathbf{x})$ *and the* $b_i(\mathbf{c},\mathbf{x})$ *are* $\mathbb{Q}$*-semipolynomials homogeneous of class* $\mathcal{C}^r$*, the* $\mathbf{c}$*-degrees of the summands in the considered equation are equal and the positivity of* $p_1(\gamma)$ *and the nonnegativity of* $p_j(\gamma)$ $(j \geq 2)$*, when* $\gamma \in W_{\mathrm{lst}}(\mathbf{R})$*, are "clearly" evident from their construction.*

∗ *The* $p_j(\mathbf{c})$ *and the* $\mathbf{x}$*-coefficients of the* $a_j(\mathbf{c},\mathbf{x})$ *and the* $b_i(\mathbf{c},\mathbf{x})$ *are semialgebraic and continuous throughout* $\mathbf{R}^{m''}$*, Nash on* $W_{\mathrm{lst}}(\mathbf{R})$ *and vanishing outside* $W_{\mathrm{lst}}(\mathbf{R})$*.*

⋄ *When* $\mathbf{R} = \mathbb{R}$*, the* $p_j(\mathbf{c})$ *and the* $\mathbf{x}$*-coefficients of the* $a_j(\mathbf{c},\mathbf{x})$ *and the* $b_i(\mathbf{c},\mathbf{x})$ *are* $\mathcal{C}^\infty$ *throughout* $\mathbb{R}^{m''}$*, analytic on* $W_{\mathrm{lst}}(\mathbb{R})$ *and vanishing outside* $W_{\mathrm{lst}}(\mathbb{R})$*.*

In order to prove these theorems we state and prove in section II a new and more general version for the Homogeneous Positivstellensatz. With respect to the other versions (see [Ste$_2$], [Del$_1$] and [Guan]) our version allows more general weights.

The main tools used in the proof of theorems A, B and C are quite similar to the techniques introduced in [DGL] to construct a continuous and rational solution of Hilbert's 17$^{\mathrm{th}}$ Problem by means of semipolynomials.

Two problems remain still unsolved: the first one, when $\mathbf{R} = \mathbb{R}$, on the existence (or not) of a $\mathcal{C}^\infty$ parametrization of Hilbert's 17$^{\mathrm{th}}$ Problem (an arbitrary real analytic parametrization was excluded by C. Delzell in [Del$_2$]).

The second one on the existence (or not) of a rational, continuous and real parametrization for the weak homogeneous complex Nullstellensatz. More precisely, we consider a polynomial system of homogeneous equations with complex coefficients

$$\mathcal{G}: \quad g_1(\mathbf{z}) = 0, \ldots, g_p(\mathbf{z}) = 0,$$

without solutions in $\mathbb{C}^n \setminus \{(0,\ldots,0)\}$. For every unknown z_i, Hilbert's Nullstellensatz provides an algebraic identity E_i in $\mathbb{C}[\mathbf{z}]$ showing that z_i is in the radical of the ideal generated by the g_j's. We fix the degrees of the polynomials g_j's and consider their coefficients as parameters

$$\mathbf{c} = (c_1, \ldots, c_{m''}) = (a_1 + \sqrt{-1}\, b_1, \ldots, a_{m''} + \sqrt{-1}\, b_{m''}) = (\mathbf{a}, \mathbf{b}).$$

The set of the parameters $(\mathbf{a}, \mathbf{b})$ such that the system $\mathcal{G}$ is impossible in $\mathbb{C}^n \setminus \{(0,\ldots,0)\}$ is an open semialgebraic set $\mathcal{S}$ in $\mathbb{R}^{2m''} \setminus \{(0,\ldots,0)\}$. Each algebraic identity E_i could have a fixed type and could be parametrized by $\mathbb{Q}$-semialgebraic continuous functions on $\mathcal{S}$.

I. Parametrizations in the non-homogeneous case.

Let $\mathbf{K}$ be an ordered field and $\mathbf{R}$ its real closure. We say that $\mathbf{K}$ is discrete when, with the point of view of constructive mathematics, the sign of any element in $\mathbf{K}$ can be determined (it is always assumed that algebraic operations in $\mathbf{K}$ are explicit). The reader interested by the theorems in classical mathematics can read the proofs in this article considering that every ordered field is discrete by the law of the excluded middle and giving no meaning to the word “explicit”. More details about this question are provided in the conclusion.

This section contains four different parts. The first one presents the general form of the polynomial Positivstellensatz, the main tool used to derive the parametrizations looked for. The second part is devoted to obtaining the semipolynomial parametrization of class $\mathcal{C}^r$, the third part to obtaining the Nash parametrization and, finally, the fourth part to obtaining, when $\mathbf{R} \subseteq \mathbb{R}$, the parametrization of class $\mathcal{C}^\infty$.

The polynomial Positivstellensatz.

First we recall the definitions of strong incompatibility and the general form for the real Nullstellensatz in the polynomial case (see [BCR], [Lom$_1$] and [Lom$_3$]) by following the notation of [Lom$_3$]. We consider an ordered field $\mathbf{K}$, and $\mathbf{x}$ denotes a list of variables $x_1, x_2, \ldots, x_n$. We then denote by $\mathbf{K}[\mathbf{x}]$ the ring $\mathbf{K}[x_1, x_2, \ldots, x_n]$. If $\mathcal{F}$ is a finite subset of $\mathbf{K}[\mathbf{x}]$, we let $\mathcal{F}^2$ be the set of squares of elements in $\mathcal{F}$, and $\mathcal{M}(\mathcal{F})$ be the *multiplicative monoid generated by* $\mathcal{F} \cup \{1\}$. $\mathcal{C}p(\mathcal{F})$ will be the *positive cone generated by* $\mathcal{F}$ (= the additive monoid generated by elements of type pPQ^2, where $0 \le p \in \mathbf{K}$, $P \in \mathcal{M}(\mathcal{F})$, and $Q \in \mathbf{K}[\mathbf{x}]$). Finally, let $I(\mathcal{F})$ be the ideal generated by $\mathcal{F}$.

Definition I.1
Consider 4 finite subsets of $\mathbf{K}[\mathbf{x}]$: $\mathbb{H}_>, \mathbb{H}_\ge, \mathbb{H}_=, \mathbb{H}_\ne$, containing polynomials for which we want respectively the sign conditions > 0, ≥ 0, $= 0$, and $\ne 0$: we say that $\mathbb{H} := [\mathbb{H}_>, \mathbb{H}_\ge, \mathbb{H}_=, \mathbb{H}_\ne]$ is *strongly incompatible* in $\mathbf{K}$ if we have in $\mathbf{K}[\mathbf{x}]$ an equality of the following type:

$$S + P + Z = 0 \quad \text{with} \quad S \in \mathcal{M}(\mathbb{H}_> \cup \mathbb{H}_\ne^2),\ P \in \mathcal{C}p(\mathbb{H}_\ge \cup \mathbb{H}_>),\ Z \in I(\mathbb{H}_=).$$

If

$$\mathbb{H}_> = \{S_1, \ldots, S_r\} \qquad \mathbb{H}_\ge = \{P_1, \ldots, P_j\} \qquad \mathbb{H}_= = \{Z_1, \ldots, Z_k\} \qquad \mathbb{H}_\ne = \{N_1, \ldots, N_h\}$$

then we use the following notation for the strong incompatibility of $\mathbb{H}$:

$$\downarrow [S_1 > 0, \ldots, S_i > 0, P_1 \ge 0, \ldots, P_j \ge 0, Z_1 = 0, \ldots, Z_k = 0, N_1 \ne 0, \ldots, N_h \ne 0] \downarrow$$

or,

$$\downarrow \mathbb{H}(x_1, \ldots, x_n) \downarrow.$$

It is clear that a strong incompatibility is a very strong form of impossibility. In particular, it implies that it is impossible to give the indicated signs to the polynomials considered, in any ordered extension of $\mathbf{K}$.

The list $\mathbb{H} := [\mathbb{H}_>, \mathbb{H}_\ge, \mathbb{H}_=, \mathbb{H}_\ne]$ appearing in the definition of strong incompatibility is called a generalized system of sign conditions on polynomials of $\mathbf{K}[\mathbf{x}]$. The different variants of the Nullstellensatz in the real case are a consequence of the following general theorem:

Theorem I.2 (Polynomial Positivstellensatz)
Let $\mathbf{K}$ be an ordered discrete field and $\mathbf{R}$ a real closed extension of $\mathbf{K}$. The three following conditions, concerning a generalized system of sign conditions on polynomials of $\mathbf{K}[\mathbf{x}]$, are equivalent:
- *strong incompatibility in $\mathbf{K}$;*
- *impossibility in $\mathbf{R}$; and*
- *impossibility in all ordered extensions of $\mathbf{K}$.*

This Nullstellensatz was first proved in 1974 [Ste$_1$]. Less general variants were given by Krivine [Kri], Dubois [Du], Prestel [Pre], Risler [Ris] and Efroymson [Efr]. All the proofs until [Lom$_1$] used the Axiom of Choice.

Semipolynomial parametrization of class $\mathcal{C}^r$.

We begin introducing a definition and several easy lemmas concerning some semipolynomials of class $\mathcal{C}^r$.

Definition I.3

Let r be a positive integer. A $\mathbf{K}$-semipolynomial f is a $\mathbf{K}$-$\mathcal{C}^r$-semipolynomial if f can be obtained as a composition (in an iterative way) of polynomial functions with coefficients in $\mathbf{K}$ and the functions:

$$\alpha \quad \longmapsto \quad (\max\{\alpha, 0\})^s$$

with $s > r$.

Remark that in the previous definition it would suffice to consider only the function:

$$\alpha \quad \longmapsto \quad (\max\{\alpha, 0\})^{r+1}$$

because the others are obtained multiplying by a convenient power of α. It is natural in this setting to ask the question (à la Pierce-Birkhoff) whether every $\mathbf{K}$-semipolynomial of class $\mathcal{C}^r$ is a $\mathbf{K}$-$\mathcal{C}^r$-semipolynomial, but the answer to this question is not necesary for our purposes.

Lemma I.4
If f is a $\mathbf{K}$-$\mathcal{C}^r$-semipolynomial then every $\mathbf{K}$-semipolynomial appearing inside the definition of f (in particular f itself) is a $\mathbf{K}$-semipolynomial of class $\mathcal{C}^r$.

Lemma I.5
Let s be an odd integer with $s > r$. The function:

$$\alpha \quad \longmapsto \quad |\alpha|^s$$

is a $\mathbb{Q}$-$\mathcal{C}^r$-semipolynomial. The graph of this function is:

$$\{(\alpha, \beta) \in \mathbf{R}^2 : \beta^2 = \alpha^{2s}, \;\; \beta \geq 0\}.$$

Lemma I.6
Let s be an odd integer with $s > r$. The functions ma_s and mi_s defined by:

$$\mathrm{ma}_s(\alpha, \beta) \stackrel{\text{def}}{=} \alpha^s + \beta^s + |\alpha - \beta|^s,$$

$$\mathrm{mi}_s(\alpha, \beta) \stackrel{\text{def}}{=} \alpha^s + \beta^s - |\alpha - \beta|^s$$

are $\mathbb{Q}$-$\mathcal{C}^r$-semipolynomials. Moreover, the following equivalences hold:

$$\mathrm{ma}_s(\alpha, \beta) > 0 \iff \alpha > 0 \quad \text{or} \quad \beta > 0,$$

$$\mathrm{ma}_s(\alpha, \beta) \geq 0 \iff \alpha \geq 0 \quad \text{or} \quad \beta \geq 0,$$

$$\mathrm{mi}_s(\alpha, \beta) > 0 \iff \alpha > 0 \quad \text{and} \quad \beta > 0,$$

$$\mathrm{mi}_s(\alpha, \beta) \geq 0 \iff \alpha \geq 0 \quad \text{and} \quad \beta \geq 0.$$

Lemma I.7
If k and r are positive integers then it is possible to construct two $\mathbb{Q}$-$\mathcal{C}^r$-semipolynomials, $\max_r$ and $\min_r$, defined on $\mathbf{R}^k$ and verifying the following equivalences:

$$\max_r(\alpha_1, \ldots, \alpha_k) > 0 \iff \alpha_1 > 0 \quad \text{or} \quad \ldots \quad \text{or} \quad \alpha_k > 0,$$

$$\max_r(\alpha_1, \ldots, \alpha_k) \geq 0 \iff \alpha_1 \geq 0 \quad \text{or} \quad \ldots \quad \text{or} \quad \alpha_k \geq 0,$$

$$\min_r(\alpha_1, \ldots, \alpha_k) > 0 \iff \alpha_1 > 0 \quad \text{and} \quad \ldots \quad \text{and} \quad \alpha_k > 0,$$

$$\min_r(\alpha_1, \ldots, \alpha_k) \geq 0 \iff \alpha_1 \geq 0 \quad \text{and} \quad \ldots \quad \text{and} \quad \alpha_k \geq 0.$$

Proof:
If $k = 2$ then $\max_r(\alpha_1, \alpha_2)$ is defined as $\mathrm{ma}_s(\alpha_1, \alpha_2)$ with s the first odd integer bigger than r. For $k > 2$, the definition of $\max_r$ is done inductively. ∎

Proposition I.8
Let r be a positive integer. Then:

- *for every closed* $\mathbf{K}$*-semialgebraic set F in* $\mathbf{R}^n$*, it is possible to construct in an explicit way a* $\mathbf{K}$-$\mathcal{C}^r$*-semipolynomial h verifying:*
$$\xi \in F \iff h(\xi) \geq 0;$$
- *for every open* $\mathbf{K}$*-semialgebraic set U in* $\mathbf{R}^n$*, it is possible to construct in an explicit way a* $\mathbf{K}$-$\mathcal{C}^r$*-semipolynomial g verifying:*
$$\xi \in U \iff g(\xi) > 0.$$

Proof:
The Finiteness Theorem (see [BCR]) allows us to describe the closed semialgebraic set F as:

$$F = \bigcup_{i=1}^{m} \{\xi \in \mathbf{R}^n : f_{i,1}(\xi) \geq 0, \ldots, f_{i,s_i}(\xi) \geq 0\},$$

with every $f_{i,j}$ a polynomial with coefficients in $\mathbf{K}$. Defining

$$h(\xi) = \max_r(\min_r(f_{1,1}(\xi), \ldots, f_{1,s_1}(\xi)), \ldots, \min_r(f_{m,1}(\xi), \ldots, f_{m,s_m}(\xi))),$$

lemma I.7 allows us to obtain the desired conclusion.

The same proof, replacing $\geq$ by $>$, applies for the open case. ∎

Next, putting together the previous propositions and the techniques introduced in [DGL], we prove the theorem analogous to theorem III.1 in [DGL], where a rational and continuous solution for some cases of the Positivstellensatz was introduced.

Theorem I.9 (Rational, $\mathcal{C}^r$ parametrization for some cases of the Positivstellensatz)
Let r be a positive integer. Let $\mathbb{H}(\mathbf{c}, \mathbf{x})$ be a generalized system of sign conditions on polynomials in $\mathbf{K}[\mathbf{c}, \mathbf{x}]$*, where the x_i's are considered as variables and the c_j's as parameters. If* $\mathbf{S}_{\mathbf{H}}(\mathbf{R})$ *is the semialgebraic set defined by*

$$\gamma \in \mathbf{S}_{\mathbf{H}}(\mathbf{R}) \iff \forall \xi \in \mathbf{R}^n \quad \mathbb{H}(\gamma, \xi) \text{ is false},$$

and if $\mathbf{S}_{\mathbf{H}}(\mathbf{R})$ *is locally closed, then there exist* $\mathbf{K}$-$\mathcal{C}^r$*-semipolynomials* $h_1(\mathbf{c})$ *and* $h_2(\mathbf{c})$ *such that*

$$\gamma \in \mathbf{S}_{\mathbf{H}}(\mathbf{R}) \iff \Big[h_1(\gamma) \geq 0, \;\; h_2(\gamma) > 0\Big].$$

If $\gamma \in \mathbf{S}_{\mathbf{H}}(\mathbf{R})$*, then the impossibility of* $\mathbb{H}(\mathbf{x}) := \mathbb{H}(\gamma, \mathbf{x})$ *inside* $\mathbf{R}^n$ *is made obvious by a strong incompatibility of fixed type (i. e. independent of γ) and with coefficients given by* $\mathbf{K}$-$\mathcal{C}^r$*-semipolynomials in* $\mathbf{c}$*. Moreover,*

- *the algebraic identity obtained, seen as a polynomial in* $\mathbf{x}$*, has an especially simple structure. More precisely, every* $\mathbf{x}$*-coefficient of this identity is a* $\mathbf{K}$-$\mathcal{C}^r$*-semipolynomial in* $\mathbf{c}$ *vanishing everywhere (without assuming $h_1(\gamma) \geq 0$ and $h_2(\gamma) > 0$), and*
- *every coefficient* $p(\mathbf{c})$ *in the algebraic identity which must be nonnegative (resp. positive) on* $\mathbf{S}_{\mathbf{H}}(\mathbf{R})$ *is given by a* $\mathbf{K}$-$\mathcal{C}^r$*-semipolynomial showing such character in an especially clear way under the hypothesis* $h_1(\mathbf{c}) \geq 0$ *and* $h_2(\mathbf{c}) > 0$*.*

Proof:
The existence of h_1 and h_2 is due to proposition I.8. The rest of the proof is identical to the proof of theorem III.1 in [DGL] with the addition of lemma I.4 assuring that every semipolynomial appearing in the proof is of class $\mathcal{C}^r$.

The proof begins by introducing the variables that appear in the straight-line programs defining the semipolynomials h_1 and h_2. Next we construct a generalized system of sign conditions considering the equations and inequalities associated to the new variables together with $h_1(\mathbf{c}) \geq 0$ and $h_2(\mathbf{c}) > 0$. The proof of the theorem is achieved applying the polynomial Positivstellensatz to this system and replacing, in the final identity obtained, every variable of the straight-line programs by the corresponding function. ∎

A corollary of the previous theorem is Theorem A, stated in the introduction, without the statements about the **c**-homogeneity of the parametrized solution (to be shown in section II). For that it is enough to consider

$$\mathbb{H}(\mathbf{c},\mathbf{x}) = [f_{n,d}(\mathbf{c},\mathbf{x}) < 0],$$

which gives $S_{\mathbf{H}}(\mathbf{R}) = \mathbb{F}_{n,d}(\mathbf{R})$.

Nash parametrization.

In this part (and in the next one) we shall deal only with the cases where the coefficients of the generalized system of sign conditions vary in an open set of the parameter space. Two interesting examples of this situation were introduced in [GL$_1$].

We begin by recalling a classical definition of a ring of functions everywhere defined and Nash on an open semialgebraic set (cf. [BCR], pages 42-43).

Definition I.10
Let U be an open semialgebraic set in $\mathbf{R}^m$. The set $\mathcal{A}(\mathbf{R}^m, \mathbf{K}, U)$ will represent the smallest subring of the ring of continuous semialgebraic functions from $\mathbf{R}^m$ to $\mathbf{R}$ containing the $\mathbf{K}$-polynomial functions and such that, if f is a sum of squares of functions in the subring, strictly positive on U, then $\sqrt{f}$ is in the subring.

Any function $f \in \mathcal{A}(\mathbf{R}^m, \mathbf{K}, U)$ can be defined by a straight-line program with the following structure. Every instruction is an assignment $t_i \longleftarrow \cdots$ with the indexes i ordered in an increasing way (the last t_i is f). The instructions can have only the two following types:

- ⋆ $t_j \longleftarrow P(x_1, \ldots, x_n, t_{i_1}, \ldots, t_{i_k})$ where $P \in \mathbf{K}[x_1, \ldots, x_n, t_{i_1}, \ldots, t_{i_k}]$ and every i_h is smaller than j,
- ⋆ $t_j \longleftarrow \sqrt{t_{i_1}^2 + \cdots + t_{i_k}^2}$ where every i_h is smaller than j and $t_{i_1}^2 + \cdots + t_{i_k}^2$ is a strictly positive function on U.

Concerning the last instruction, the value of t_j can be characterized by the following generalized system of sign conditions:

$$t_j^2 - (t_{i_1}^2 + \cdots + t_{i_k}^2) = 0, \quad t_j \geq 0.$$

It is worthwhile to remark in this point that, in the case where $\mathbf{K}$ is discrete and the open semialgebraic set U is given in an explicit way, there exists an explicit test to decide if a straight-line program such as the one shown before is correct, i.e., if every instruction of the second type is right.

The next theorem provides a way of defining an open semialgebraic set by means of a Nash function.

Theorem I.11
If U is an open $\mathbf{K}$-semialgebraic set in $\mathbf{R}^m$ then there exists a function $f \in \mathcal{A}(\mathbf{R}^m, \mathbf{K}, U)$ strictly positive on U and vanishing outside U.

The proof of this theorem given in [BCR] is fully constructive in the case where $\mathbf{K}$ is an ordered discrete field and the open semialgebraic set U is given in an explicit way. Moreover, the theorem in [BCR] is stated for the case in which $\mathbf{K} = \mathbf{R}$ (i.e., with $\mathcal{A}(\mathbf{R}^m, \mathbf{R}, U)$), but in fact, the proof shows the rational version of the theorem (as in [BCR]'s proofs of some of the previously mentioned theorems, including, for example, the Finiteness Theorem).

Theorem I.12 (Nash parametrization for some cases of the Positivstellensatz)
Let $\mathbb{H}(\mathbf{c},\mathbf{x})$ be a generalized system of sign conditions on polynomials in $\mathbf{K}[\mathbf{c},\mathbf{x}]$, where the x_i's are considered as variables and the c_j's as parameters. Let $\mathbf{S}_{\mathbf{H}}(\mathbf{R})$ be the semialgebraic set defined by

$$\gamma \in \mathbf{S}_{\mathbf{H}}(\mathbf{R}) \iff \forall \xi \in \mathbf{R}^n \quad \mathbb{H}(\gamma, \xi) \text{ is false,}$$

and let us assume that $\mathbf{S}_{\mathbf{H}}(\mathbf{R})$ is open. If $\gamma \in \mathbf{S}_{\mathbf{H}}(\mathbf{R})$ then the impossibility of $\mathbb{H}(\mathbf{x}) := \mathbb{H}(\gamma, \mathbf{x})$ inside $\mathbf{R}^n$ is made obvious by a strong incompatibility of fixed type (independent of γ) and with "coefficients" given by functions in $\mathbf{c}$ belonging to $\mathcal{A}(\mathbf{R}^m, \mathbf{K}, \mathbf{S}_{\mathbf{H}}(\mathbf{R}))$ and vanishing outside $\mathbf{S}_{\mathbf{H}}(\mathbf{R})$. In particular, they are Nash functions on the open set $\mathbf{S}_{\mathbf{H}}(\mathbf{R})$, and if $\mathbf{K}$ is a real 2-closed field (i.e., if every positive element in $\mathbf{K}$ has a square root in $\mathbf{K}$) then they take values in $\mathbf{K}$, for the points in $\mathbf{S}_{\mathbf{H}}(\mathbf{R})$ with coordinates in $\mathbf{K}$.

Proof:
Using the proof of theorem III.1 in [DGL] together theorem I.11 we obtain an algebraic identity with the following type:

$$p_1(\mathbf{c})^s S(\mathbf{c},\mathbf{x}) + \sum_j a_j(\mathbf{c}) Q_j(\mathbf{c},\mathbf{x}) v_j(\mathbf{c},\mathbf{x})^2 + \sum_j N_j(\mathbf{c},\mathbf{x}) w_j(\mathbf{c},\mathbf{x}) = 0.$$

Let $\mathbb{H}$ be equal to $[\mathbb{H}_>, \mathbb{H}_\geq, \mathbb{H}_=, \mathbb{H}_\neq]$. The polynomial $S(\mathbf{c},\mathbf{x})$ is a product of polynomials in $\mathbb{H}_> \cup \mathbb{H}_\neq^2$, the $Q_j(\mathbf{c},\mathbf{x})$ are products of polynomials in $\mathbb{H}_> \cup \mathbb{H}_\geq$ and the $N_j(\mathbf{c},\mathbf{x})$ are polynomials in $\mathbb{H}_=$. The function $p_1 \in \mathcal{A}(\mathbf{R}^m, \mathbf{K}, \mathbf{S_H}(\mathbf{R}))$ is strictly positive on $\mathbf{S_H}(\mathbf{R})$ and vanishes outside $\mathbf{S_H}(\mathbf{R})$. The $a_j(\mathbf{c})$, nonnegative on $\mathbf{S_H}(\mathbf{R})$, the $\mathbf{x}$-coefficients of the $v_j(\mathbf{c},\mathbf{x})$ and the $w_j(\mathbf{c},\mathbf{x})$ are also in the ring $\mathcal{A}(\mathbf{R}^m, \mathbf{K}, \mathbf{S_H}(\mathbf{R}))$.

To obtain all the requirements in the theorem it suffices to show that all the functions in $\mathbf{c}$ introduced vanish outside $\mathbf{S_H}(\mathbf{R})$. To achieve this goal we use the function $p_1(\mathbf{c})$ as multiplier: multiplying, in the previous equation, the first term by $p_1(\mathbf{c})^3$, every $a_j(\mathbf{c})$ by $p_1(\mathbf{c})$, every coefficient of each $v_j(\mathbf{c})$ by $p_1(\mathbf{c})$ and every coefficient of each $w_j(\mathbf{c})$ by $p_1(\mathbf{c})^3$, we obtain the equality looked for. ∎

Parametrization of class $\mathcal{C}^\infty$.

The next theorem is more surprising than the theorems in the previous parametrizations, due to its non semialgebraic character. As the field of real numbers (which has no an explicit sign test) appears here in an unavoidable way, a discussion about the constructive nature of the theorem is needed and will be given in section IV.

Theorem I.13 ($\mathcal{C}^\infty$ parametrization for some cases of the Positivstellensatz for $\mathbb{R}$)
Let $\mathbf{K}$ be a discrete subfield of $\mathbb{R}$ and $\mathbf{R}$ the real closure of $\mathbf{K}$. Let $\mathbb{H}(\mathbf{c},\mathbf{x})$ be a generalized system of sign conditions on polynomials in $\mathbf{K}[\mathbf{c},\mathbf{x}]$, where the x_i's are considered as variables and the c_j's as parameters. Let $\mathbf{S_H}(\mathbf{R})$ be the semialgebraic set defined by

$$\gamma \in \mathbf{S_H}(\mathbf{R}) \iff \forall \xi \in \mathbf{R}^n \quad \mathbb{H}(\gamma, \xi) \text{ is false,}$$

and let us assume that $\mathbf{S_H}(\mathbf{R})$ is open. If $\gamma \in \mathbf{S_H}(\mathbf{R})$ then the impossibility of $\mathbb{H}(\mathbf{x}) := \mathbb{H}(\gamma,\mathbf{x})$ inside $\mathbf{R}^n$ is made obvious by a strong incompatibility of fixed type (independent of γ) and with "coefficients" given by functions in $\mathbf{c}$ of class $\mathcal{C}^\infty$, analytic on $\mathbf{S_H}(\mathbb{R})$ and vanishing outside $\mathbf{S_H}(\mathbb{R})$.

Proof:
The proof is obtained using the same arguments as in the proof of theorem I.12, with the only difference in the choice of the multiplier. Defining:

$$\eta(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ e^{-1/t} & \text{if } t > 0 \end{cases}$$

the multiplier providing the proof of the theorem is

$$\mu(\mathbf{c}) = \eta(p_1(\mathbf{c})),$$

where $p_1(\mathbf{c})$ is the multiplier used in the proof of theorem I.12. The proof of the fact that $\mu(\mathbf{c})$ is a function of class $\mathcal{C}^\infty$, and the same for $\mu(\mathbf{c}) \cdot q(\mathbf{c})$ when $q(\mathbf{c})$ is a Nash function on $\mathbf{S_H}(\mathbb{R})$ (semialgebraic and continuous on $\mathbf{R}^n$), is easy and left to the reader. ∎

II. Homogeneous versions of the Positivstellensatz and other theorems in Real Algebraic Geometry.

Let $\mathbf{K}$ be an ordered field and $\mathbf{R}$ its real closure. This section begins introducing the homogeneous setting we shall consider. Let ℓ be a fixed integer and $\mathbf{x}$ a set of variables (some of them will be considered as parameters sometimes). To every variable in $\mathbf{x}$ is assigned a weight: a list of ℓ nonnegative rational numbers (usually these rational numbers are integers and $\ell = 1$ or $\ell = 2$). The weight of a monomial is defined, as usual, to be the sum of the weights of the variables occuring in it (counted with multiplicity).

The set of degrees, or weights, of the monomials is the subset $\mathcal{W}$ of $\mathbb{Q}_+^\ell$ generated (by addition) by the weights of the variables. This set is provided with a total order relation $\preceq$ satisfying the following properties:
-. $\preceq$ is compatible with the addition,
-. $\alpha_1 \leq \beta_1, \ldots, \alpha_\ell \leq \beta_\ell \quad \Rightarrow \quad (\alpha_1, \ldots, \alpha_\ell) \preceq (\beta_1, \ldots, \beta_\ell)$,
-. $\mathcal{W}$ and any finitely generated additive monoid in $\mathbb{Q}_+^\ell$ is well–ordered by $\preceq$.
In this setting, a polynomial is homogeneous if all its monomials have the same weight.

A Homogeneous Positivstellensatz.

First, the definition of strong incompatibility is extended to the homogeneous case.

Definition II.1
Let $\mathbf{K}$ be an ordered field. Consider a strong incompatibility for a generalized system of sign conditions $\mathbb{H}$ over homogeneous polynomials:

$$\downarrow [S_1 > 0, \ldots, S_r > 0, P_1 \geq 0, \ldots, P_j \geq 0, Z_1 = 0, \ldots, Z_k = 0, N_1 \neq 0, \ldots, N_h \neq 0] \downarrow$$

$$\mathbb{H}_> = \{S_1, \ldots, S_r\} \qquad \mathbb{H}_\geq = \{P_1, \ldots, P_j\} \qquad \mathbb{H}_= = \{Z_1, \ldots, Z_k\} \qquad \mathbb{H}_\neq = \{N_1, \ldots, N_h\}$$

with the structure

$$S + \sum_i \delta_i A_i B_i^2 + Z_1 C_1 + \cdots + Z_k C_k = 0 \qquad (\star)$$

where $S \in \mathcal{M}(\mathbb{H}_> \cup \mathbb{H}_\neq^2)$, the δ_i are positive elements in $\mathbf{K}$, every A_i belongs to $\mathcal{M}(\mathbb{H}_\geq \cup \mathbb{H}_>)$ and the B_i and C_j are polynomials in $\mathbf{K}[\mathbf{x}]$. The strong incompatibility $\mathbb{H}$ is called **homogeneous** if all the polynomials in $(\star)$ are homogeneous and if all the summands in $(\star)$ have the same degree. It will be denoted in the following way:

$$\downarrow [S_1 > 0, \ldots, S_r > 0, P_1 \geq 0, \ldots, P_j \geq 0, Z_1 = 0, \ldots, Z_k = 0, N_1 \neq 0, \ldots, N_h \neq 0] \downarrow_{\text{homogeneous}}.$$

Using the ideas in the proof introduced by G. Stengle in [Ste$_2$], we obtain a general homogeneous Positivstellensatz as an algorithmic consequence of the polynomial Positivstellensatz.

Theorem II.2 (Homogeneous Positivstellensatz)
Let $\mathbf{K}$ be an ordered discrete field and $\mathbf{R}$ a real closed extension of $\mathbf{K}$. The three following conditions, concerning a generalized system of sign conditions on homogeneous polynomials of $\mathbf{K}[\mathbf{x}]$, are equivalent:
- *the existence of a homogeneous strong incompatibility in $\mathbf{K}$;*
- *impossibility in $\mathbf{R}$; and*
- *impossibility in all ordered extensions of $\mathbf{K}$.*

Proof:
First we consider the case of the Nullstellensatz: several equalities and one inequality of type $\neq$. The proof of the theorem will be obtained by means of a trick “à la Rabinowitsch” over the Nullstellensatz case.

In the Nullstellensatz case we deal with the generalized system of sign conditions:

$$\mathbb{H}: \quad Z_1 = 0, \ldots, Z_k = 0, N \neq 0.$$

Applying the polynomial Positivstellensatz we obtain an algebraic identity with the following structure:

$$N^{2s} + \sum_{i=1}^{t} \delta_i B_i^2 + Z_1 C_1 + \cdots + Z_k C_k = 0 \qquad (1)$$

with $s \in \mathbb{N}$, $0 < \delta_i \in \mathbf{K}$ and $B_i, C_i \in \mathbf{K}[\mathbf{x}]$.

Let $2p$ be the weight of N^{2s}. If every polynomial B_i^2 and $Z_j C_j$ has no monomials with weight smaller than $2p$, then the desired homogeneous strong incompatibility is obtained by replacing every B_i and every

C_j by their homogeneous parts B'_i and C'_j of suitable weights so that the resulting summands $\delta_i B'^2_i$ and $Z_j C'_j$ have weights $2p$.

The remaining case is solved by deleting the homogeneous parts with weight smaller than $2p$ in the following way. Defining $B_0 = N^s$ and $\delta_0 = 1$, equation (1) becomes:

$$\sum_{i=0}^{t} \delta_i B_i^2 + Z_1 C_1 + \cdots + Z_k C_k = 0 \tag{2}$$

with B_0 homogeneous.

If q is a weight, then $B_{i,q}$ will denote the homogeneous part of B_i with weight q and $C_{u,q}$ the homogeneous part of C_u such that $Z_u C_{u,q}$ is the homogeneous part of $Z_u C_u$ with weight $2q$. Using these definitions, the identity (2) provides, for every weight q, a homogeneous identity with the following structure:

$$\sum_{i=0}^{t} \delta_i B_{i,q}^2 + 2 \sum_{i=0}^{t} \delta_i \sum_{\substack{u+v=2q \\ u<q}} B_{i,u} B_{i,v} + Z_1 C_{1,q} + \cdots + Z_k C_{k,q} = 0 \tag{3,q}$$

Next, by induction on those q ocurring as weights in the B_i's, it is proved that every $B_{i,q}$ satisfies a homogeneous strong incompatibility:

$$\downarrow [Z_1 = 0, \ldots, Z_k = 0, B_{i,q} \neq 0] \downarrow_{\text{homogeneous}} \tag{4,i,q}$$

A polynomial D is said to satisfy a homogeneous strong incompatibility of type (4) when

$$\downarrow [Z_1 = 0, \ldots, Z_k = 0, D \neq 0] \downarrow_{\text{homogeneous}}.$$

Then the homogeneous strong incompatibilities $(4, i, q)$ are obtained from the equations in $(3, q)$ and the following easy–to–derive stability properties (see [Ste$_2$] or [Del$_1$] for details):

- if A and B are homogeneous polynomials with the same degree satisfying homogeneous strong incompatibilities of type (4) then $A + B$ also has this property,
- if A and B are homogeneous polynomials and A satisfies a homogeneous strong incompatibility of type (4) then AB also has this property,
- if B is a homogeneous polynomials satisfying a homogeneous strong incompatibility of type (4), and A is a homogeneous polynomial satisfying a homogeneous identity of type

$$A^2 + B + \sum_{i=0}^{t} \delta_i V_i^2 + Z_1 D_1 + \cdots + Z_k D_k = 0,$$

  then A satisfies a homogeneous strong incompatibility of type (4).

To prove the general case it can be assumed without loss of generality that the generalized system of sign conditions on homogeneous polynomials with which we are dealing is:

$$\mathbb{H}: \quad P_1 \geq 0, \ldots, P_r \geq 0, Z_1 = 0, \ldots, Z_k = 0, N \neq 0.$$

The proof will be obtained proving by induction on r that if the considered system is impossible in $\mathbf{R}$ then there exists a homogeneous strong incompatibility in $\mathbf{K}$:

$$\downarrow [P_1 \geq 0, \ldots, P_r \geq 0, Z_1 = 0, \ldots, Z_k = 0, N \neq 0] \downarrow_{\text{homogeneous}}.$$

The technique "à la Rabinowitsch", next explained, corresponds, following the terminology in [Lom$_1$] or [Lom$_3$], to the potential existence of the square root of a positive element, which is established here in the homogeneous setting.

The case $r = 0$ has already been proved: it is the Nullstellensatz case previously considered. The inductive step is performed by introducing a new variable y with weight equal to '$(\deg P_r)/2$', and considering the impossible (over $\mathbf{R}$) generalized system of sign conditions on homogeneous polynomials:

$$\mathbb{K}: \quad P_1 \geq 0, \ldots, P_{r-1} \geq 0, y^2 - P_r = 0, Z_1 = 0, \ldots, Z_k = 0, N \neq 0.$$

Using the inductive hypothesis we obtain a homogeneous strong incompatibility:

$$\downarrow [P_1 \geq 0, \ldots, P_{r-1} \geq 0, y^2 - P_r = 0, Z_1 = 0, \ldots, Z_k = 0, N \neq 0] \downarrow_{\text{homogeneous}}$$

that corresponds to the explicit algebraic identity:

$$S + \sum_i \delta_i A_i B_i^2 + (y^2 - P_r)C + Z_1 C_1 + \cdots + Z_k C_k = 0$$

where $S = N^{2s}$ $(s \in \mathbb{N})$, $0 < \delta_i \in \mathbf{K}$, $A_i \in \mathcal{M}(P_1, \ldots, P_{r-1}) \subset \mathbf{K}[\mathbf{x}]$ and $B_i, C_i, C \in \mathbf{K}[\mathbf{x}, y]$.

The homogeneous identities:

$$y^{2m} - P_r^m = (y^2 - P_r)(y^{2m-2} + y^{2m-2}P_r + \cdots + y^2 P_r^{m-2} + P_r^{m-1})$$

allow us to replace, in the B_i and C_i, every y^{2m} by P_r^m and every y^{2m+1} by yP_r^m, obtaining a new homogeneous identity with the following structure :

$$S + \sum_i \delta_i A_i (E_i + yF_i)^2 + (y^2 - P_r)G + Z_1(G_1 + yH_1) + \cdots + Z_k(G_k + yH_k) = 0$$

with $E_i, F_i, G_i, H_i \in \mathbf{K}[\mathbf{x}]$ and $G \in \mathbf{K}[\mathbf{x}, y]$. Developing the squares and replacing $y^2 F_i^2$ by $P_r F_i^2$, after grouping the y terms, we obtain a new homogeneous identity:

$$S + \sum_i \delta_i A_i E_i^2 + \sum_i \delta_i A_i P_r F_i^2 + (y^2 - P_r)H + Z_1 G_1 + \cdots + Z_k G_k + yK = 0$$

with $K \in \mathbf{K}[\mathbf{x}]$ and $H \in \mathbf{K}[\mathbf{x}, y]$. This last identity implies that $H = 0$ (and $K = 0$): otherwise considering H as a polynomial in y, its leading coefficient is the leading coefficient (with respect y) of the left hand side of the previous identity, which implies directly its vanishing. This last assertion provides the desired homogeneous identity (without y):

$$S + \sum_i \delta_i A_i E_i^2 + \sum_i \delta_i A_i P_r F_i^2 + Z_1 G_1 + \cdots + Z_k G_k = 0$$

and the proof of the theorem. ∎

The homogeneous Positivsllensatz obtained in the previous theorem is more general than the versions presented in [Ste_2], [Del_1] or [Guan], because of the use of more general weights. In [Ste_2] all the variables have weight equal to 1, and in [Del_1] and [Guan] the weight of some variables is equal to 1 and for the others is equal to 0. Our homogeneous version, making the distinction between variables and parameters as in theorem B ($\star$), does not appear in [Guan]. In fact all the proofs are based on the algorithm shown in [Ste_1] (even the one in [Guan]).

A Projective Finiteness Theorem.

This section is devoted to presenting a Finiteness Theorem for projective spaces. We remark that a homogeneous Finiteness Theorem more general than the one to be presented in this section can be probably stated (and proved), but we restrict our attention only to the cases appearing in the applications we are going to obtain. This theorem will be only applied to a semialgebraic set in the coefficient space which will be considered as a projective space in the usual natural way.

The real projective space, $\mathbb{P}_u(\mathbf{R})$, can be regarded as a closed and bounded affine real algebraic variety (see [BCR] chapter 3). To every vectorial line Δ in $\mathbf{R}^{u+1}$ is assigned the matrix Π_Δ corresponding to the orthogonal projection on Δ. This matrix is an element of the affine $\mathbb{Q}$-algebraic set:

$$\mathbf{Q}_u(\mathbf{R}) = \{M \in \mathbf{M}_{u+1}(\mathbf{R}) : M = M^t, \;\; M^2 = M, \;\; \mathrm{Trace}(M) = 1\}.$$

This provides a biregular isomorphism Ψ between $\mathbb{P}_u(\mathbf{R})$ and $\mathbf{Q}_u(\mathbf{R})$. If $\mathbf{z}$ is a nonzero vector in Δ then the matrix $\Psi(\Delta) = \Phi(\mathbf{z}) = \mathbf{y}$ has in the (i,j)–position:

$$y_{i,j} = \frac{(z_i z_j)}{\|\mathbf{z}\|^2}.$$

Semialgebraic sets, euclidean topology and semialgebraic functions on $\mathbb{P}_u(\mathbf{R})$ are well defined: see, for example, [BCR] chapter 3. It is also very easy to verify that any open semialgebraic set in $\mathbb{P}_u(\mathbf{R})$ can be characterized by the corresponding semialgebraic open set in $\mathbf{R}^{u+1} \setminus \{(0,\ldots,0)\}$ or by the corresponding open semialgebraic set in the unit sphere. The bijection Ψ is also an isomorphism for these semialgebraic notions.

Definition II.3 (Basic projective semialgebraic sets)
A closed (resp. open) $\mathbf{K}$–semialgebraic set in the projective space $\mathbb{P}_u(\mathbf{R})$, regarded as the corresponding semialgebraic set in $\mathbf{R}^{u+1} \setminus \{(0,\ldots,0)\}$, is a basic projective $\mathbf{K}$–semialgebraic set if it can be described as a finite intersection of closed (resp. open) sets with the following type:

$$\{\zeta \in \mathbf{R}^{u+1} \setminus \{(0,\ldots,0)\} : S(\zeta) \geq \text{(resp. } >) \; 0\},$$

where S is a homogeneous polynomial with even degree in $\mathbf{K}[\mathbf{z}]$.

Let F be a closed $\mathbf{K}$–semialgebraic set in $\mathbf{Q}_u(\mathbf{R}) \subset \mathbf{R}^{(u+1)^2}$. Using the Finiteness Theorem, the set F can be described as a finite union of basic closed $\mathbf{K}$–semialgebraic sets:

$$F = \bigcup_{k=1}^{h} \bigcap_{\ell=1}^{n_k} \{\eta \in \mathbf{R}^{(u+1)^2} : R_{k,\ell}(\eta) \geq 0\}$$

where $R_{k,\ell}(\mathbf{y}) \in \mathbf{K}[\mathbf{y}]$ and $\mathbf{y} = (y_{i,j})_{1 \leq i,j \leq u+1}$.

For every polynomial $R(\mathbf{y})$ of degree d, the polynomial $\Phi^\bullet(R)$ is defined as follows:

$$\Phi^\bullet(R)(\mathbf{z}) = \|\mathbf{z}\|^{2d} R(\Phi(\mathbf{z})).$$

Clearly the polynomial $\Phi^\bullet(R)$ is homogeneous with even degree and allows us to describe the closed set G in $\mathbf{R}^{u+1} \setminus \{(0,\ldots,0)\}$ corresponding to $\Psi^{-1}(F)$ in the following terms:

$$G = \bigcup_{k=1}^{h} \bigcap_{\ell=1}^{n_k} \{\zeta \in \mathbf{R}^{u+1} \setminus \{(0,\ldots,0)\} : \Phi^\bullet(R_{k,\ell})(\zeta) \geq 0\}$$

where the $\Phi^\bullet(R_{k,\ell})(\mathbf{z})$ are even degree homogeneous polynomials in $\in \mathbf{K}[\mathbf{z}]$.

Using the same arguments for the open case we have obtained the proof of our projective Finiteness Theorem.

Theorem II.4 (Projective Finiteness Theorem)
Every closed (resp. open) $\mathbf{K}$–semialgebraic set in the projective space $\mathbb{P}_u(\mathbf{R})$, regarded as the corresponding semialgebraic set in $\mathbf{R}^{u+1} \setminus \{(0,\ldots,0)\}$, can be described as a finite union of basic closed (resp. open) projective $\mathbf{K}$–semialgebraic sets.

Remark II.5
All the homogeneous polynomials involved in the description of a closed semialgebraic set in $\mathbb{P}_u(\mathbf{R})$ as a finite union of basic closed projective semialgebraic sets can be chosen with the same degree: it suffices to multiply those with smaller degree by a convenient power of the polynomial $z_1^2 + \cdots + z_{u+1}^2$.

Homogeneous semialgebraic functions (case of projective spaces).

Projective space $\mathbb{P}_u(\mathbf{R})$ is a non singular real $\mathbb{Q}$-algebraic variety. This implies that the rational, regular and Nash functions are well defined on $\mathbb{P}_u(\mathbf{R})$.

Homogeneous polynomials with even degree, $P\colon \mathbf{R}^{u+1} \to \mathbf{R}$, allow us to define, through its restriction to the sphere, a particular family of regular functions on $\mathbb{P}_u(\mathbf{R})$. These functions agree with the $\Phi^\bullet(R)$ introduced in the previous section and they are a subring: if P and Q do not have the same degree then, to obtain $P+Q$, it suffices to multiply the one with the smaller degree by a convenient power of $z_1^2+\cdots+z_{u+1}^2$ before performing the addition. Nevertheless this subring can not be defined in an intrinsic way.

Definition II.6
Let q be a nonnegative integer. A semialgebraic function

$$f\colon \begin{array}{ccc} \mathbf{R}^{u+1} & \longrightarrow & \mathbf{R} \\ \zeta & \longmapsto & f(\zeta) \end{array}$$

is said to be **homogeneous** with weight q if

$$\forall \lambda \in \mathbf{R} \qquad f(\lambda \cdot \zeta) = \lambda^q f(\zeta).$$

When q is even, the restriction of such a function to the unit sphere defines a semialgebraic function on $\mathbb{P}_u(\mathbf{R})$.

We shall be especially interested in the case when the function f can be defined by means of a straight-line program where all the assignments are "homogeneous" and rendering evident the properties of f we need. A first case, where this situation appears, corresponds to some semipolynomial functions.

Definition II.7
An even homogeneous $\mathbf{K}$-$\mathcal{C}^r$-semipolynomial expression f is a straight-line program where every instruction is a "homogeneous" assignment $t_i \longleftarrow \ldots$ allowing us to give, without ambiguity, an even weight to the variable t_i. More precisely, the indexes i are ordered in a strictly increasing way (the last t_i defines f) and the instructions can have only the two following types:

- ⋆ $t_j \longleftarrow P(z_1, \ldots, z_{u+1}, t_{i_1}, \ldots, t_{i_k})$ where $P \in \mathbf{K}[z_1, \ldots, z_{u+1}, t_{i_1}, \ldots, t_{i_k}]$ is homogeneous with even weight and every i_h is smaller than j,
- ⋆ $t_j \longleftarrow |t_i|^s$ with s odd, $s > r$ and $i < j$.

We remark that t_1 is a homogeneous polynomial in the variables z_i.

As in the non-homogeneous case, the two following facts, concerning an even homogeneous $\mathbf{K}$-$\mathcal{C}^r$-semipolynomial expression, are true:

- any variable t_i inside the program defines a $\mathbf{K}$-$\mathcal{C}^r$-semipolynomial that is homogeneous with even degree and of class $\mathcal{C}^r$,
- the assignments in the program can be characterized by systems of equations and inequations.

It is easy to show that the homogeneous version of lemma I.7 is true. More precisely, using our projective Finiteness Theorem and remark II.5 it is possible to state the analogous statements to I.7 and I.8.

Proposition II.8
1-. *If k and r are positive integers and $f_1, \ldots, f_k$ are even homogeneous $\mathbf{K}$-$\mathcal{C}^r$-semipolynomial expressions, then it is possible to construct in an explicit way two even homogeneous $\mathbf{K}$-$\mathcal{C}^r$-semipolynomial*

expressions, g and h, verifying the following equivalences:

$$g > 0 \iff f_1 > 0 \quad \text{or} \quad \dots \quad \text{or} \quad f_k > 0,$$

$$g \geq 0 \iff f_1 \geq 0 \quad \text{or} \quad \dots \quad \text{or} \quad f_k \geq 0,$$

$$h > 0 \iff f_1 > 0 \quad \text{and} \quad \dots \quad \text{and} \quad f_k > 0,$$

$$h \geq 0 \iff f_1 \geq 0 \quad \text{and} \quad \dots \quad \text{and} \quad f_k \geq 0.$$

2-. *If r is a positive integer and F a homogeneous closed $\mathbf{K}$-semialgebraic set (regarded as a closed semialgebraic set in $\mathbf{R}^{u+1} \setminus \{(0,\dots,0)\}$), then it is possible to construct in an explicit way an even homogeneous $\mathbf{K}$-$\mathcal{C}^r$-semipolynomial expression h satisfying:*

$$\zeta \in F \iff h(\zeta) \geq 0.$$

3-. *Statements similar to (2) for open and locally closed sets are true.*

Next we study the straight-line programs defining some particular homogeneous Nash functions.

Definition II.9
Let U be an open semialgebraic set in $\mathbf{R}^{u+1} \setminus \{(0,\dots,0)\}$ saturated by the equivalence relation $\equiv$ defining in the usual way the projective u-dimensional space. A function $f \in \mathcal{A}(\mathbf{R}^{u+1}, \mathbf{K}, U)$ is said to be homogeneous with even weight if its definition is made through a straight-line-program where every instruction is a homogeneous assignment $t_i \longleftarrow \dots$ allowing us to give an even weight to the variable t_i. More precisely, the indexes i are ordered in a strictly increasing way (the last t_i defines f) and the instructions can have only the two following types:

- ⋆ $t_j \longleftarrow P(z_1,\dots,z_{u+1},t_{i_1},\dots,t_{i_k})$, where $P \in \mathbf{K}[z_1,\dots,z_{u+1},t_{i_1},\dots,t_{i_k}]$ is homogeneous with even weight and every i_h is smaller than j,
- ⋆ $t_j \longleftarrow \sqrt{t_{i_1}^2 + \cdots + t_{i_k}^2}$ with every i_h smaller than j, the t_{i_h} with the same weight (which is assigned to t_j) and $t_{i_1}^2 + \cdots + t_{i_k}^2$ defining a strictly positive function on U.

The functions verifying the conditions in the last definition are a well defined class of Nash functions on the projective space $\mathbb{P}_u(\mathbf{R})$. The same arguments used to prove our projective Finiteness Theorem provide a Nash version for the projective open semialgebraic sets. The details are left to the reader, using remark II.5 to equalize degrees when needed.

Theorem II.10
If U is an open semialgebraic set in $\mathbf{R}^{u+1} \setminus \{(0,\dots,0)\}$ saturated by the equivalence relation $\equiv$, then there exists a function $f \in \mathcal{A}(\mathbf{R}^{u+1}, \mathbf{K}, U)$ strictly positive on U, vanishing outside U and defined as homogeneous with even weight.

In [GLM] it is proved that every integral semialgebraic continuous function can be described by a straight-line-program using as instructions polynomials with coefficients in $\mathbf{K}$ and some elementary root functions (which are continuous and $\mathbb{Q}$-semialgebraic). The proof used there could probably be adapted to the homogeneous general case providing more general definitions and using the fact that any root function of a monic degree d polynomial

$$(a_{d-1},\dots,a_0) \longmapsto \rho_\sigma(a_{d-1},\dots,a_0)$$

is homogeneous with weight p if every a_i is homogeneous with weight $(d-i)p$.

III. Parametrizations in the homogeneous case.

Let $\mathbf{K}$ be an ordered field and $\mathbf{R}$ its real closure. The theorems for the homogeneous case shown in section II will allow us to obtain the homogeneous versions of the parametrization theorems proved in section I. *In what follows the space of coefficients is always a projective space.*

Homogeneous $\mathcal{C}^r$ Parametrization.

Theorem III.1 (Rational, homogeneous, $\mathcal{C}^r$ parametrization for some cases of the Positivstellensatz)
Let r be a positive integer. We give weights to the variables x_i and to the parameters c_j. All the c_j have the same non–zero weight, independent of the x_i's weights (for example the weight of every x_i could be $(r_i, 0)$ with r_i a nonnegative rational and the weight of every c_j could be $(0, 1)$)[1]. *Let $\mathbb{H}(\mathbf{c}, \mathbf{x})$ be a generalized system of sign conditions on homogeneous polynomials in $\mathbf{K}[\mathbf{c}, \mathbf{x}]$. Let $\mathbf{S_H}(\mathbf{R})$ be the semialgebraic set defined by*

$$\gamma \in \mathbf{S_H}(\mathbf{R}) \iff \forall \xi \in \mathbf{R}^n \quad \mathbb{H}(\gamma, \xi) \text{ is false},$$

and let us assume that $\mathbf{S_H}(\mathbf{R})$ is a locally closed projective set. If $\gamma \in \mathbf{S_H}(\mathbf{R})$ then the impossibility of $\mathbb{H}(\mathbf{x}) := \mathbb{H}(\gamma, \mathbf{x})$ inside $\mathbf{R}^n$ is made obvious by a strong incompatibility of fixed type (independent of γ) and with coefficients given by $\mathbf{K}$-$\mathcal{C}^r$-semipolynomials homogeneous with even weight in $\mathbf{c}$ and with all the terms in the sum homogeneous with the same degree in $\mathbf{c}$.

Proof:
As a consequence of proposition II.8 and theorem II.2, it is enough to use the same proof presented for I.9. It is also possible to introduce in the homogeneous case the same refinements presented in I.9 for the non-homogeneous case. The independence between the weights of the variables and the parameters guarantees that $\mathbf{S_H}(\mathbf{R})$ is a cone (i.e., an union of rays): this is the reason why we need the projective hypothesis on $\mathbf{S_H}(\mathbf{R})$. ∎

Next we prove the homogeneous statements in Theorem A. The existence of the linear form $h_{n,d}(\mathbf{c})$ is due to the following fact: for each $\eta \in \mathbb{Z}^n \setminus \{(0, \ldots, 0)\}$, the linear form in $\mathbf{c}$, $f_{n,d}(\mathbf{c}, \eta)$, is nonnegative if $\mathbf{c} = \gamma \in \mathbb{F}_{n,d}(\mathbf{R})$. If we consider a finite set of points in $\mathbb{Z}^n \setminus \{(0, \ldots, 0)\}$ such that the corresponding linear forms are a basis of the dual space, then $h_{n,d}(\mathbf{c})$ can be defined as the sum of these linear forms.

Now we consider the generalized system of sign conditions

$$\mathbb{H}(\mathbf{c}, \mathbf{x}) = [h_{n,d}(\mathbf{c}) f_{n,d}(\mathbf{c}, \mathbf{x}) < 0].$$

The saturated closed set $\mathbb{F}_{n,d}(\mathbf{R}) \cup -\mathbb{F}_{n,d}(\mathbf{R})$ is strictly contained in $\mathbf{S_H}(\mathbf{R})$. Let $u_{n,d}(\mathbf{c})$ be an even homogeneous $\mathbf{K}$-$\mathcal{C}^r$-semipolynomial expression satisfying

$$\forall \gamma \in \mathbf{R}^m \quad \Big(u_{n,d}(\gamma) \geq 0 \iff \gamma \in \mathbb{F}_{n,d}(\mathbf{R}) \cup -\mathbb{F}_{n,d}(\mathbf{R}) \Big).$$

Then we consider the following impossible generalized system of sign conditions

$$\mathbb{K}(\mathbf{c}, \mathbf{x}) = [u_{n,d}(\mathbf{c}) \geq 0,\ h_{n,d}(\mathbf{c}) f_{n,d}(\mathbf{c}, \mathbf{x}) < 0].$$

Giving the weight 0 to the x_i's and the weight 1 to the c_j's and reasoning as in theorems I.9 and III.1, we get the complete proof of the Theorem A in the introduction.

To prove the item $\star$ in Theorem B, we use the generalized system of sign conditions

$$\mathbb{H}(\mathbf{c}, \mathbf{x}) = [c_1 g_{n,d}(\mathbf{c}, \mathbf{x}) \leq 0].$$

In this case $\mathbf{S_H}(\mathbf{R}) = \mathbb{U}_{n,d}(\mathbf{R}) \cup -\mathbb{U}_{n,d}(\mathbf{R})$. The desired proof is obtained by giving the weight $(0, 1)$ to the x_i's and the weight $(1, 0)$ to the c_j's and applying Theorem III.1.

To prove the item $\star$ in Theorem C, we use the generalized system of sign conditions

$$\mathbb{H}(\mathbf{c}, \mathbf{x}) = [g_1(\mathbf{c}, \mathbf{x}) = 0,\ \ldots,\ g_p(\mathbf{c}, \mathbf{x}) = 0].$$

In this case $\mathbf{S_H}(\mathbf{R}) = W_{\text{lst}}(\mathbf{R})$. The desired proof is obtained by giving the weight $(0, 1)$ to the x_i's and the weight $(1, 0)$ to the c_j's and applying Theorem III.1.

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1 More precisely, the weights of the x_i are independent of the weights of the c_j if the intersection of the two corresponding generated subspaces over $\mathbb{Q}$ is trivial.

Homogeneous Nash Parametrization.

Theorem III.2 (Homogeneous Nash parametrization for some cases of the Positivstellensatz)
We give weights to the variables x_i and to the parameters c_j. All the c_j have the same non–zero weight, independent of the x_i's weights. Let $\mathbb{H}(\mathbf{c},\mathbf{x})$ be a generalized system of sign conditions on homogeneous polynomials in $\mathbf{K}[\mathbf{c},\mathbf{x}]$. Let $\mathbf{S_H}(\mathbf{R})$ be the semialgebraic set defined by

$$\gamma \in \mathbf{S_H}(\mathbf{R}) \iff \forall \xi \in \mathbf{R}^n \quad \mathbb{H}(\gamma,\xi) \text{ is false},$$

and let us assume that $\mathbf{S_H}(\mathbf{R})$ is an open projective set on $m{+}1$ variables. If $\gamma \in \mathbf{S_H}(\mathbf{R})$ then the impossibility of $\mathbb{H}(\mathbf{x}) := \mathbb{H}(\gamma,\mathbf{x})$ inside $\mathbf{R}^n$ is made obvious by a strong incompatibility of fixed type (independent of γ) and with coefficients given by homogeneous functions with even degree in $\mathcal{A}(\mathbf{R}^{m+1},\mathbf{K},\mathbf{S_H}(\mathbf{R}))$ vanishing outside of $\mathbf{S_H}(\mathbf{R})$, all the summands being homogeneous and with the same degree in $\mathbf{c}$. In particular, these functions are Nash on the open set $\mathbf{S_H}(\mathbf{R})$ and if $\mathbf{K}$ is real 2-closed (every positive element in $\mathbf{K}$ is a square) then they send $\mathbf{K}^{m+1}$ into $\mathbf{K}$.

Giving $(0,1)$ as weight for every variable x_i and $(1,0)$ for every parameter c_j, the previous theorem provides as particular cases the items $(*)$ in theorems B and C in the introduction.

Homogeneous $\mathcal{C}^\infty$ Parametrization.

Theorem III.3 (Homogeneous $\mathcal{C}^\infty$ parametrization for some cases of the Positivstellensatz for $\mathbb{R}$)
Let $\mathbf{K}$ be a discrete subfield of $\mathbb{R}$. We give weights to the variables x_i and to the parameters c_j. All the c_j have the same non–zero weight, independent of the x_i's weights. Let $\mathbb{H}(\mathbf{c},\mathbf{x})$ be a generalized system of sign conditions on homogeneous polynomials in $\mathbf{K}[\mathbf{c},\mathbf{x}]$. Let $\mathbf{S_H}(\mathbf{R})$ be the semialgebraic set defined by

$$\gamma \in \mathbf{S_H}(\mathbf{R}) \iff \forall \xi \in \mathbf{R}^n \quad \mathbb{H}(\gamma,\xi) \text{ is false},$$

and let us assume that $\mathbf{S_H}(\mathbf{R})$ is an open projective set. If $\gamma \in \mathbf{S_H}(\mathbf{R})$ then the impossibility of $\mathbb{H}(\mathbf{x}) := \mathbb{H}(\gamma,\mathbf{x})$ inside $\mathbf{R}^n$ is made obvious by a strong incompatibility of fixed type (independent of γ) and with coefficients given by functions in $\mathbf{c}$ of class $\mathcal{C}^\infty$, analytic on $\mathbf{S_H}(\mathbf{R})$ and vanishing outside $\mathbf{S_H}(\mathbf{R})$.

Giving $(0,1)$ as weight for every variable x_i and $(1,0)$ for every parameter c_j, the previous theorem provides as particular cases the items $(\diamond)$ in theorems B and C in the introduction.

IV. Conclusions: the constructive content of the results.

In Constructive Mathematics (see [BB] or [MRR]), the theorems presented in the previous sections are true for any discrete ordered field $\mathbf{K}$ and its real closure $\mathbf{R}$ (see [LR]), because in this setting we have a constructive proof of the Positivstellensatz (see [Lom_1]).

This paper has been written from the point of view of a constructive mathematician (see [BB] or [MRR]). Anyway it can be read as a paper in classical mathematics where all the proofs are effective, in particular without using the Axiom of Choice or the law of the excluded middle, providing primitive recursive algorithms (in case of primitive recursive discrete real closed fields) or uniformly primitive recursive ones (in case of discrete real closed fields, see [LR], e.g., if the structure of coefficient field is given by an oracle giving the sign of any polynomial with integer coefficients evaluated in the coefficients of the problem).

It seems also important to study the constructive content of the presented results when working with the field of the real numbers, $\mathbb{R}$, in Constructive Analysis (see [BB]). Such real numbers are defined as (explicitly) Cauchy sequences of rational numbers but, at every moment, we only know a finite number of terms in these sequences. This discussion is really needed for the $\mathcal{C}^\infty$ parametrization theorems where the real numbers appear in an unavoidable way.

From an algorithmic point of view this means that the coefficients appearing in the theorem, the parameters in $\mathbf{c}$ (if we are in a parametrization theorem) and the variables $\mathbf{x}$ are given by oracles providing suitable rational approximations (depending on what has been asked to the oracle) of these real numbers and that we look for uniform algorithms (which, in general, will be uniformly primitive recursive). The proofs of the theorems presented in this article do not provide automatically such algorithms, i.e., in this case the proofs are not constructive because in $\mathbb{R}$ we have no sign tests. Anyway these theorems could have a constructive version, especially in the case where the semialgebraic set $\mathbf{S_H}(\mathbb{R})$ is defined on a discrete subfield of $\mathbb{R}$. For example, this happens in the 17^{th} Hilbert's Problem already considered in [DGL] and [GL_1].

The results presented in this section will be discussed in a more detailed form in [GL_2]. We assume, in any of the parametrization theorems, that the polynomials in $(\mathbf{c}, \mathbf{x})$ defining the system $\mathbb{H}$ have their coefficients inside some discrete subfield $\mathbf{K}$ of $\mathbb{R}$. This is the case, for example, in theorems A, B, and C in the introduction. Let $\mathbf{R}$ be the real closure of $\mathbf{K}$. Clearly we have in a constructive way the following equivalence:

$$\mathbb{H}(\gamma, \xi) \text{ is false for every } \xi \in \mathbf{R}^n \iff \gamma \in \mathbf{S_H}(\mathbf{R}) \iff h_1(\gamma) > 0 \quad \text{and} \quad h_2(\gamma) \geq 0$$

where h_1 and h_2 are two continuous $\mathbf{K}$-semialgebraic functions.

If $\mathbf{S_H}(\mathbf{R})$ is closed (resp. open) then we can write $h_1 = 1$ (resp. $h_2 = 1$) or delete the sign condition $h_1(\mathbf{c}) > 0$ (resp. $h_2(\mathbf{c}) \geq 0$). Let $\mathbf{S_H}(\mathbb{R})$ be the set defined by:

$$\mathbf{S_H}(\mathbb{R}) = \{\gamma \in \mathbb{R}^m : h_1(\gamma) > 0,\ h_2(\gamma) \geq 0\}.$$

So defined, $\mathbf{S_H}(\mathbb{R})$ depends, a priori, on $\mathbf{S_H}(\mathbf{R})$ and also on the semialgebraic functions h_1 and h_2. It can be proved that the actual dependence is only on $\mathbf{S_H}(\mathbf{R})$ due to the fact that some easy cases of the classical transfer principle are still true in a constructive setting (see [GL_2]).

First we prove that if $\gamma \in \mathbf{S_H}(\mathbb{R})$ then the algebraic identity constructed implies that the generalized system of sign conditions $\mathbb{H}$ is false for every $\xi \in \mathbb{R}^n$. In fact the system $\mathbb{H}$ will be impossible under the strong constructive form:

$$\begin{array}{c}\text{conjuntion of}\\ \text{strict sign conditions}\end{array} \Longrightarrow \begin{array}{c}\text{constructive disjunction of}\\ \text{strong negations of non strict sign conditions}\end{array}$$

This is shown by assuming, without loss of generality, that $\mathbb{H}$ is the system:

$$A(\mathbf{c}, \mathbf{x}) \neq 0,\ \ B_1(\mathbf{c}, \mathbf{x}) \geq 0, \ldots, B_r(\mathbf{c}, \mathbf{x}) \geq 0.$$

With the constructed algebraic identity, we have the implication (where the "$\vee$" are constructive):

$$\forall \gamma \in \mathbb{R}^m\ \forall \xi \in \mathbb{R}^n \quad \Big(\big(h_1(\gamma) > 0 \wedge h_2(\gamma) \geq 0 \wedge A(\gamma, \xi) \neq 0\big) \Longrightarrow \big(B_1(\gamma, \xi) < 0 \vee \ldots \vee B_r(\gamma, \xi) < 0\big)\Big)$$

which can be read in the following terms:

$$\forall \gamma \in \mathbf{S_H}(\mathbb{R}) \quad \forall \xi \in \mathbb{R}^n \quad [A(\gamma, \xi) \neq 0 \implies B_1(\gamma, \xi) < 0 \vee \cdots \vee B_r(\gamma, \xi) < 0]$$

as we wanted to show.

Finally it should be necessary to discuss, case by case, how the condition $\gamma \in \mathbf{S_H}(\mathbb{R})$ is implied (in the constructive way) by the impossibility of $\mathbb{H}(\gamma, \mathbf{x})$ in $\mathbb{R}^n$. Here we consider only the converse in the case of Theorem B. We have just proved the constructive implication:

$$\forall \gamma \in \mathbb{R}^m \quad \Big(h(\gamma) > 0 \quad \Longrightarrow \quad \big(\forall \xi \in \mathbb{R}^n \setminus \{(0, \ldots, 0)\} \quad g_{n,d}(\gamma, \xi) > 0\big)\Big)$$

with $h(\mathbf{c})$ a well defined continuous $\mathbb{Q}$-semialgebraic function. The goal to be achieved is the constructive proof of the converse presented in the following form:

$$\forall \gamma \in \mathbb{R}^m \quad \Big(\big(\forall \xi \in \mathbb{R}^n \setminus \{(0, \ldots, 0)\} \quad g_{n,d}(\gamma, \xi) > 0\big) \quad \Longrightarrow \quad h(\gamma) > 0\Big).$$

First we restrict our attention to the compact spheres

$$S_m(\mathbb{R}) = \{\gamma \in \mathbb{R}^m : \|\gamma\| = 1\} \qquad S_n(\mathbb{R}) = \{\xi \in \mathbb{R}^n : \|\xi\| = 1\}$$

and define the functions:

$$f(\mathbf{c}) = \sup\{0, h(\mathbf{c})\} \qquad\qquad k(\mathbf{c}) = \sup\{0, \inf\{g_{n,d}(\mathbf{c}, \xi) : \xi \in S_n(\mathbb{R})\}\}.$$

Since $S_n\mathbb{R}$ is compact, the function $k(\mathbf{c})$ is continuous and well defined on $\mathbb{R}^m$, f and k being the extensions by continuity of their restrictions to the real algebraic numbers which can be obtained by the methods of discrete Real Algebraic Geometry.

The zeroes of f are contained in the zeroes of k in the discrete case. This fact allows us to obtain a Łojasievicz Inequality (hidden in the parametrized Positivstellensatz):

$$\forall \gamma \in S_m(\mathbf{R}) \qquad k(\gamma)^p \leq a \cdot f(\gamma)$$

with a a positive rational. This non-strict inequality is extended by continuity to $\mathbb{R}^m$ which provides a constructive proof of the implication:

$$\forall \gamma \in \mathbb{R}^m \quad \Big(k(\gamma) > 0 \quad \Longrightarrow \quad f(\gamma) > 0 \Big)$$

and for the implication:

$$\forall \gamma \in \mathbb{R}^m \setminus \{(0, \ldots, 0)\} \quad \Big(\inf\{g_{n,d}(\gamma, \xi) : \xi \in S_n(\mathbb{R})\} > 0 \quad \Longrightarrow \quad h(\gamma) > 0 \Big).$$

The desired converse will be fully proved if we are able to prove constructively the implication:

$$\forall \gamma \in S_m(\mathbb{R}) \qquad \Big(\big(\forall \xi \in S_n(\mathbb{R}) \quad g_{n,d}(\gamma, \xi) > 0 \big) \quad \Longrightarrow \quad \inf\{g_{n,d}(\gamma, \xi) : \xi \in S_n(\mathbb{R})\} > 0 \Big)$$

i.e., a polynomial with real coefficients strictly positive on $S_n(\mathbb{R})$ is lower bounded on $S_n(\mathbb{R})$ by a strictly positive real number.

To clarify the constructive meaning of this kind of results, true in the classical setting, is one of the objectives of [GL$_2$]. For the reader not yet convinced by this philosophy, let us remark that this question has a precise and incontestable mathematical meaning: to find an algorithm computing a strictly positive lower bound for a polynomial with real coefficients on the sphere knowing that such polynomial is strictly positive on the sphere.

A first algorithm appears in a natural way: since the γ_i's are supposed known through oracles providing suitable rational approximations then for every integer k it is computed with precision $1/2^k$ a lower bound for $g_{n,d}(\mathbf{c}, \mathbf{x})$ on the sphere $S_n(\mathbb{R})$. This process will stop when the result of the computation assures that this lower bound is strictly positive (of course after a finite, but not determined, number of steps).

The reader convinced by this algorithm has arrived to the conclusion that Theorem B has been proved constructively for the field of the real numbers "á la Cauchy". Anyway, taking a stricter constructive point of view, as in [BB], this algorithm does not solve fully the problem because it is not possible to estimate its computing time: we have no constructive proof for the termination of the algorithm. But if we are able of reducing the computation of the minimum on the sphere to the computation of the minimum on a finite number of points, then the constructive proof will be obtained.

Nevertheless it is well known that there exists no general constructive proof of the classical theorem assuring that any uniformly continuous and strictly positive function on a compact set is lower bounded by a strictly positive real number. This impossibility comes from the fact that one can compute a recursive (in a reasonable sense) and uniformly continuous function which takes its minimum, zero, only in non recursive points, of a compact interval (see for example [Bee], Theorem 9.1, pp. 73).

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