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Commutative algebra. Constructive methods. Finite type projective modules. (Algèbre commutative. Méthodes constructives. Modules projectifs de type fini. Cours et exercices.) (French)

Mathématiques en Devenir. Paris: Calvage et Mounet. xxxi, 991 p. EUR 69.00 (2011). ISBN 978-2-91-635221-3/pbk

This voluminous book offers a very special and detailed introduction to various basic concepts, methods, principles, and results of commutative algebra. In contrast to most of the numerous other primers in the field, the authors consequently pursue the constructive viewpoint in commutative algebra, that is, they endeavour to extricate the explicit algorithmic approaches to the many existence theorems in this central area of contemporary mathematical research, thereby revisiting several classical abstract topics in a highly clarifying, remarkably simplifying, and methodologically novel way. Very much in the spirit of the great developers of constructive algebra in the 19th century, first and foremost C. F. Gauss and L. Kronecker, the authors reveal the algorithmic aspects of such naturally abstract topics as Galois theory, Dedekind rings, Prüfer rings, finitely generated projective modules, dimension theory of commutative rings, and others in the current treatise, and that in a very enlightening, inspiring and truly unique style of expository presentation. In particular, a special characteristic of the text is the strong emphasis on the study of finitely generated projective modules in commutative algebra, which actually play a significant role as an algebraic version of the notion of vector bundle in different areas of modern geometry.

Assuming a basic knowledge of linear algebra, group theory, elementary number theory as well as the fundamentals of ring and module theory, the book under review is mainly geared toward graduate students on the levels M1 and M2 at European universities, but also researchers, instructors, and theoretical computer scientists can profit a great deal from the novel ideas and aspects exposed in this rich source of modern constructive algebra.

As for the precise contents, the book consists of seventeen chapters and an appendix on constructive logic. Each chapter is divided into several thematic sections, with a set of related exercises and some bibliographic comments at the end.

After a very carefully composed preface, including a rather detailed description of the contens of the single chapters of the book, Chapter 1 gives some motivating geometric examples, basically the concepts of vector bundles and modules of differential forms. Chapter 2 explains the "local-global principle" in commutative algebra, together with its relations to systems of linear equations over commutative rings. Chapter 3 develops the so-called "method of indeterminate coefficients" initiated by Gauss, based on which several existence theorems are then constructively treated. Modules of finite presentation are introduced in Chapter 4 where also special rings, Fitting ideals, and resultant ideals are discussed. Chapter 5 gives a first approach to finitely generated projective modules, while Chapter 6 deals with strictly finite algebras and Galois algebras, thereby providing some constructive basic Galois theory as well as fundamental results on étale

algebras and separable algebras. Chapter 7 presents another constructive method in (polynomial) algebra, namely the so-called "dynamic method". By this method, the abstract proofs of various existence theorems in algebra can be made explicit, ranging from Hilbert's Nullstellensatz to Galois theory over discrete fields. Chapter 8 turns to flat modules, with particular emphasis on flat ideals, flat algebras, and faithfully flat algebras. Local rings and their relatives are the main topic of Chapter 9 where the constructive aspects of decomposable rings as well as concrete examples of local rings in algebraic geometry are discussed along the way.
Chapter 10 gives a more in-depth treatment of finitely generated modules, together with their relations to algebro-geometric constructions such as Grassmannians, Grothendieck groups, Picard groups, and others. Chapter 11 depicts the appearance of distributive lattices and reticular groups in the context of constructive commutative algebra, and the subsequent Chapter 12 provides a very explicit and versatile description of the structure theory of Prüfer rings and of Dedekind rings. The notion of Krull dimension of commutative rings is dealt with in Chapter 13. This includes a constructive definition of that concept just as a description, of integral ring extensions, dimension theory of morphisms, valuative dimension theory, and a Krull dimension theory of distributive lattices. Chapter 14 explains the fundamental results concerning the classical problem of determining the number of generators of a finitely generated module. The authors give constructive, fairly elementary versions of the relevant theorems due to L. Kronecker, H.Bass, J.-P.Serre, O. Forster, and R. Swan, respectively, along with a discussion of the so-called Heitmann dimension of a commutative ring. Chapter 15 presents a number of important constructive methods which are directly related to the local-global principle in commutative algebra, and Chapter 16 turns to the study of finitely generated projective modules over a polynomial ring. The authors present here constructive approaches to the fundamental results by Traverso-Swan, Quillen-Vaserstein, Horrocks, Suslin, Coquand, Lombardi, Quitté, and others in a very detailed manner, thereby using their own research contributions effectively. Chapter 17 is devoted to an entirely constructive proof of A. Suslin's famous stability theorem in the case of a discrete ground field, where the concrete local-global principle of *R. Rao* (1985) appears as a crucial ingredient.
Apart from an instructive appendix on some principles of constructive logic, a rich bibliography of 187 references, a table of theorems provedtin the book, and two, very carefully compiled indices of both notions and terms provide an utmost useful service to the reader. The incredible wealth of more than 350 exercises and problems represent another outstanding feature of this marvelous, weighty and rather unique textbook in commutative algebra. In fact, these well-arranged exercises, together with their helpful hints for solution, form an essential component of the book, which provides an abundance of additional important material and examples at the same time.
All together, the book under review must be seen as an invaluable replenishment of the existing textbook literature in commutative algebra.

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Keywords : textbook (commutative algebra); constructive algebra; computational algebra; projective modules; local-global principle; algebras
Classification :
∗13-01 Textbooks (commutative rings and algebras)
13C10 Projective modules, etc.

13C11 Injective modules etc.
13F05 Dedekind and Pruefer rings and their generalizations
13C15 Dimension theory, etc. (commutative rings)
13P99 Computational aspects of commutative algebra