# A dynamical comparison between the rings $\mathbf{R}(X)$ and $\mathbf{R}\langle X\rangle$

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**Abstract**

We prove that for any ring $\mathbf{R}$ with Krull dimension $\leq d$, the ring $\mathbf{R}\langle X\rangle$ "dynamically behaves like the ring $\mathbf{R}(X)$ or a localization of a polynomial ring of type $\mathbf{R}_S[X]$ where $S$ is a multiplicatively closed subset of $\mathbf{R}$ and $\operatorname{Kdim} \mathbf{R}_S \leq d-1$" .

As application, we give a simple and constructive proof of the Lequain-Simis Theorem which is an important variation of the Quillen Induction Theorem.

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## Introduction

In this paper, we continue to follow the philosophy developed in the papers [1, 3, 4, 5, 7, 9, 14, 15, 16, 17, 18, 19, 20, 21, 23, 24, 27, 33, 34]. The main goal is to find the constructive content hidden in abstract proofs of concrete theorems.

The general method consists in replacing some abstract ideal objects whose existence is based on the third excluded middle principle and the axiom of choice by incomplete specifications of these objects. We think that this is a first step in the achievement of Hilbert's program for abstract algebra methods.

Let $\mathbf{R}$ be a commutative unitary ring. We denote by $S$ (respectively, $U$) the multiplicative subset of $\mathbf{R}[X]$ formed by monic polynomials (respectively, primitive polynomials, i.e., polynomials whose coefficients generate the whole ring). Let

$$\mathbf{R}\langle X\rangle := S^{-1}\mathbf{R}[X] \text{ and } \mathbf{R}(X) := U^{-1}\mathbf{R}[X].$$

The interest in the properties of $\mathbf{R}\langle X\rangle$ and $\mathbf{R}(X)$ branched in many directions and is attested by the abundance of articles on $\mathbf{R}\langle X\rangle$ and $\mathbf{R}(X)$ appearing in the literature (see [8] for a comprehensive list of papers dealing with the rings $\mathbf{R}\langle X\rangle$ and $\mathbf{R}(X)$). The ring $\mathbf{R}\langle X\rangle$ played an important role in Quillen's solution to Serre's conjecture [28] and its succeeding generalizations to non-Noetherian rings [2, 13, 25]. The construction $\mathbf{R}(X)$ turned out to be an efficient tool for proving results on $\mathbf{R}$ via passage to $\mathbf{R}(X)$.

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It is clear that we have $\mathbf{R}[X] \subseteq \mathbf{R}\langle X\rangle \subseteq \mathbf{R}(X)$ and that $\mathbf{R}(X)$ is a localization of $\mathbf{R}\langle X\rangle$. The containment $\mathbf{R}\langle X\rangle \subseteq \mathbf{R}(X)$ becomes an equality if and only if $\operatorname{Kdim}\mathbf{R} = 0$ [10].

In this paper, we will prove that for any ring $\mathbf{R}$ with Krull dimension $\leq d$, the ring $\mathbf{R}\langle X\rangle$ "dynamically behaves like the ring $\mathbf{R}(X)$ or a localization of a polynomial ring of type $\mathbf{R}[\frac{1}{a}][X]$ with $a \in \mathbf{R}$ and $\operatorname{Kdim}\mathbf{R}[\frac{1}{a}] \leq d-1$" . As application of our dynamical comparison between the rings $\mathbf{R}(X)$ and $\mathbf{R}\langle X\rangle$, we give a constructive variation of Lequain-Simis Induction Theorem below using a simple proof. Note that Lequain and Simis put considerable effort for proving this marvellous theorem and they used some quite complicated technical steps. Coupled with a result by Simis and Vasconcelos [31] asserting that over a valuation ring $\mathbf{V}$, all projective $\mathbf{V}[X]$-modules are free, they obtain that for any Prüfer domain $\mathbf{R}$, all finitely generated projective $\mathbf{R}[X_1,\dots,X_n]$-module are extended from $\mathbf{R}$.

**Lequain-Simis Induction Theorem** *Suppose that a class of rings $\mathcal{F}$ satisfies the following properties:*

(i) *If $\mathbf{R} \in \mathcal{F}$, then every nonmaximal prime ideal of $\mathbf{R}$ has finite height.*

(ii) $\mathbf{R} \in \mathcal{F} \Rightarrow \mathbf{R}[X]_{\mathfrak{p}[X]} \in \mathcal{F}$ *for any prime ideal $\mathfrak{p}$ of $\mathbf{R}$.*

(iii) $\mathbf{R} \in \mathcal{F} \Rightarrow \mathbf{R}_{\mathfrak{p}} \in \mathcal{F}$ *for any prime ideal $\mathfrak{p}$ of $\mathbf{R}$.*

(iv) $\mathbf{R} \in \mathcal{F}$ *and* $\mathbf{R}$ *local* $\Rightarrow$ *any finitely generated projective module over $\mathbf{R}[X]$ is free.*

*Then, for each $\mathbf{R} \in \mathcal{F}$, if $M$ is a finitely generated projective $\mathbf{R}[X_1,\dots,X_n]$-module, then $M$ is extended from $\mathbf{R}$.*

**Constructive Induction Theorem** *Let $\mathcal{F}$ be a class of commutative rings with finite Krull dimensions satisfying the properties below:*

(ii') *If $\mathbf{R} \in \mathcal{F}$ then $\mathbf{R}(X) \in \mathcal{F}$.*

(iii) $\mathbf{R} \in \mathcal{F} \Rightarrow \mathbf{R}[\frac{1}{s}] \in \mathcal{F}$ *for each $s$ in $\mathbf{R}$.*

(iv') *If $\mathbf{R} \in \mathcal{F}$ then any finitely generated projective module over $\mathbf{R}[X]$ is extended from $\mathbf{R}$.*

*Then, for each $\mathbf{R} \in \mathcal{F}$, if $M$ is a finitely generated projective $\mathbf{R}[X_1,\dots,X_n]$-module, then $M$ is extended from $\mathbf{R}$.*

For the purpose to prepare the ground for the generalizations of the Quillen-Suslin theorem quoted above, we will give in Section 2 a constructive proof of a non-Noetherian version of the Quillen-Suslin theorem for zero-dimensional rings. As a mater of fact, we will prove constructively that for any zero-dimensional ring $\mathbf{R}$, all finitely generated projective $\mathbf{R}[X_1,\dots,X_n]$-modules of constant rank are free (the Quillen-Suslin theorem corresponding to the particular case $\mathbf{R}$ is a field). Note that there is no such constructive proof in the literature.

The undefined terminology is standard as in [11, 12], and, for constructive algebra in [22, 26].

# 1 A dynamical comparison between the rings $\mathbf{R}(X)$ and $\mathbf{R}\langle X\rangle$

## 1.1 Constructive preliminaries

If $S$ is a multiplicative subset of a ring $\mathbf{R}$, the localization of $\mathbf{R}$ at $S$ is the ring $S^{-1}\mathbf{R} = \{\frac{x}{s}, x \in \mathbf{R}, s \in S\}$ in which the elements of $S$ are forced into being invertible. For $x_1,\dots,x_r \in \mathbf{R}$, $\mathcal{M}(x_1,\dots,x_r)$ will denote the multiplicative subset of $\mathbf{R}$ generated by $x_1,\dots,x_r$, that is,

$$\mathcal{M}(x_1,\dots,x_r) = \{x_1^{n_1}\cdots x_r^{n_r}, n_i \in \mathbb{N}\}.$$

The localization of $\mathbf{R}$ at $\mathcal{M}(x_1,\ldots,x_r)$ is the same one as the localization at $\mathcal{M}(x_1\cdots x_r)$. If $x\in\mathbf{R}$, the localization of $\mathbf{R}$ at the multiplicative subset $\mathcal{M}(x)$ will be denoted by $\mathbf{R}_x$.

**Definition 1.1** *(Comaximal multiplicative subsets [7])*
*If $S_1,\ldots,S_k$ are multiplicative subsets of $\mathbf{R}$, we say that $S_1,\ldots,S_k$ are* comaximal *if*

$$\forall s_1\in S_1,\ldots,s_n\in S_n,\ \exists\, a_1,\ldots,a_n\in\mathbf{R}\ \text{such that}\sum_{i=1}^{n}a_is_i=1.$$

Remark that comaximal multiplicative sets remain comaximal when you replace the ring by a bigger one or the multiplicative subsets by smaller ones.

**Definition 1.2** *(Constructive definition of the radical)*
*Constructively, the* radical $\mathrm{Rad}(\mathbf{R})$ *of a ring $\mathbf{R}$ is the set of all the $x\in\mathbf{R}$ such that $1+x\mathbf{R}\subset\mathbf{R}^\times$, where $\mathbf{R}^\times$ is the group of units of $\mathbf{R}$. A ring $\mathbf{R}$ is* local *if it satisfies:*

$$\forall x\in\mathbf{R}\quad x\in\mathbf{R}^\times\ \vee\ 1+x\in\mathbf{R}^\times. \tag{1}$$

*It is* residually discrete local *if it satisfies:*

$$\forall x\in\mathbf{R}\quad x\in\mathbf{R}^\times\ \vee\ x\in\mathrm{Rad}(\mathbf{R}) \tag{2}$$

From a classical point of view, we have (1) $\Leftrightarrow$ (2), but the constructive meaning of (2) is stronger than that of (1). Constructively a *discrete field* is defined as a ring in which each element is zero or invertible, with an explicit test for the "or". An *Heyting field* (or a field) is defined as a local ring whose Jacobson radical is 0. So $\mathbf{R}$ is residually discrete local exactly when it is local and the residue field $\mathbf{R}/\mathrm{Rad}(\mathbf{R})$ is a discrete field.

**Definition 1.3** *(Constructive definition of Krull dimension [3, 6, 16])*
*A ring $\mathbf{R}$ is said to have Krull dimension less or equal to $d$ (in short, $\mathrm{Kdim}\,\mathbf{R}\le d$) if for every $x\in\mathbf{R}$, $\mathrm{Kdim}\,S_x^{-1}\mathbf{R}\le d-1$, where $S_x=\{x^k(1+yx),k\in\mathbb{N},y\in\mathbf{R}\}$ and with the initialization $\mathrm{Kdim}\,\mathbf{R}\le -1$ if $1=0$ in $\mathbf{R}$ ($\mathbf{R}$ is trivial). A ring $\mathbf{R}$ is said to be finite-dimensional if $\mathrm{Kdim}\,\mathbf{R}\le d$ for some $d\in\mathbb{N}$.*

As a particular case, if $\mathrm{Kdim}\,\mathbf{R}\le d$, $d\ge 0$ and $x\in\mathrm{Rad}(\mathbf{R})$ then, constructively, $\mathrm{Kdim}\,\mathbf{R}[1/x]\le d-1$.

Classically an arithmetical ring is a ring in which finitely generated ideals are locally principal. This is equivalent to the following constructive definition.

**Definition 1.4** *(Constructive definition of arithmetical rings [7])*
*A ring $\mathbf{R}$ is said to be* arithmetical *if for any $a,b\in\mathbf{R}$ there is an $u$ such that $\langle a,b\rangle=\langle a\rangle$ in $\mathbf{R}_u$ and $\langle a,b\rangle=\langle b\rangle$ in $\mathbf{R}_{1-u}$. A* Prüfer ring *is a reduced arithmetical ring. An* integral domain *is a ring in which each element is zero or regular (with an explicit test for the "or"). A* Prüfer domain *is an arithmetical ring which is an integral domain. A* valuation domain *is a local Prüfer domain.*

Valuations domains are also characterized as integral domains such that for any $a,b$, $a$ divides $b$ or $b$ divides $a$ (with an explicit test for the "or" and an explicit divisibility for "divides").

**The General Constructive Rereading Principle**

Let us now recall a General Constructive Rereading Principle which enables to automatically obtain a "quasi-global" version of a theorem from its local version.

Let $I$ and $U$ be two subsets of $\mathbf{R}$. We denote by $\mathcal{M}(U)$ the monoid generated by $U$, $\mathcal{I}_{\mathbf{R}}(I)$ or $\mathcal{I}(I)$ the ideal generated by $I$, and $\mathcal{S}_{\mathbf{R}}(I;U)$ or $\mathcal{S}(I;U)$ the monoid $\mathcal{M}(U)+\mathcal{I}_{\mathbf{R}}(I)$. If $I=\{a_1,\ldots,a_k\}$ and $U=\{u_1,\ldots,u_\ell\}$, we denote $\mathcal{M}(U)$, $\mathcal{I}(I)$ and $\mathcal{S}(I;U)$ respectively by $\mathcal{M}(u_1,\ldots,u_\ell)$, $\mathcal{I}(a_1,\ldots,a_k)$ and $\mathcal{S}(a_1,\ldots,a_k;u_1,\ldots,u_\ell)$.

Note that in the ring $S^{-1}\mathbf{R}$, where $S=\mathcal{S}(a_1,\ldots,a_k;u_1,\ldots,u_\ell)$, the $u_j$'s are invertible and the $a_i$'s are in the Jacobson radical. Moreover it is easy to see that for any $a\in\mathbf{R}$, the monoids $\mathcal{S}(I;U,a)$ and $\mathcal{S}(I,a;U)$ are comaximal in $\mathbf{R}_{\mathcal{S}(I;U)}$. These two remarks lead to the desired rereading principle.

**General Principle 5 of** [21] *When rereading an explicit proof given in case* $\mathbf{R}$ *is residually discrete local, with an arbitrary ring* $\mathbf{R}$*, start with* $\mathbf{R}=\mathbf{R}_{\mathcal{S}(0;1)}$*. Then, at each disjunction (for an element* $a$ *produced when computing in the local case)*

$$a\in\mathbf{R}^\times\ \vee\ a\in\mathrm{Rad}(\mathbf{R}),$$

*replace the "current" ring* $\mathbf{R}_{\mathcal{S}(I;U)}$ *by both* $\mathbf{R}_{\mathcal{S}(I;U,a)}$ *and* $\mathbf{R}_{\mathcal{S}(I,a;U)}$ *in which the computations can be pursued. At the end of this rereading, one obtains a finite family of rings* $\mathbf{R}_{\mathcal{S}(I_j;U_j)}$ *with comaximal monoids* $\mathcal{S}(I_j;U_j)$ *and finite sets* $I_j$, $U_j$.

## 1.2 The rings $\mathbf{R}(X)$ and $\mathbf{R}\langle X\rangle$

By the following theorem, we prove that for any ring $\mathbf{R}$ with Krull dimension $\le d$, the ring $\mathbf{R}\langle X\rangle$ "dynamically behaves like the ring $\mathbf{R}(X)$ or a localization of a polynomial ring of type $\mathbf{R}[\frac{1}{a}][X]$ with $a\in\mathbf{R}$ and $\mathrm{Kdim}\,\mathbf{R}[\frac{1}{a}]\le d-1$".

**Theorem 1.5** *Let* $d\in\mathbb{N}$ *and* $\mathbf{R}$ *a ring with Krull dimension* $\le d$*. Then for any primitive polynomial* $f\in\mathbf{R}[X]$*, there exist comaximal subsets* $\mathcal{M}_1,\ldots,\mathcal{M}_s$ *of* $\mathbf{R}\langle X\rangle$ *such that for each* $1\le i\le s$*, either* $f$ *is invertible in* $\mathbf{R}\langle X\rangle_{\mathcal{M}_i}$ *or* $\mathbf{R}\langle X\rangle_{\mathcal{M}_i}$ *is a localization of* $\mathbf{R}_{S_i}[X]$ *for some monoid* $S_i\subseteq\mathbf{R}$ *such that* $\mathbf{R}_{S_i}$ *has Krull dimension* $\le d-1$.

**Proof**

*First case:* $\mathbf{R}$ *is residually discrete local.* Observe that any primitive polynomial $f\in\mathbf{R}[X]$ can be written in the form $f=g+u$ where $g$, $u\in\mathbf{R}[X]$, all the coefficients of $g$ are in the Jacobson radical $\mathrm{Rad}(\mathbf{R})$ of $\mathbf{R}$ and $u$ is quasi monic (that is, the leading coefficient of $u$ is invertible). If the degree of $u$ is $k$, then $g=\sum_{j>k}a_jX^j$. Now we open two branches: we localize $\mathbf{R}\langle X\rangle$ at the comaximal multiplicative subsets generated by $f$ and $g$:

$$\begin{array}{ccc} & \mathbf{R}\langle X\rangle & \\ \swarrow & & \searrow \\ \mathbf{R}\langle X\rangle_f & & \mathbf{R}\langle X\rangle_g \end{array}$$

In $\mathbf{R}\langle X\rangle_f$, $f$ is clearly invertible.

In $\mathbf{R}\langle X\rangle_g$, write $g=\sum_{j=k+1}^m a_jX^j$, where the $a_j\in\mathrm{Rad}(\mathbf{R})$. It follows that the multiplicative subsets $\mathcal{M}(a_{k+1}),\ldots,\mathcal{M}(a_s)$ are comaximal in $\mathbf{R}\langle X\rangle_g$. Note that for any $k+1\le i\le m$, $\mathcal{M}(a_i)^{-1}(\mathbf{R}\langle X\rangle_g)$ is a localization of the polynomial ring $\mathbf{R}_{a_i}[X]$ and $\dim\mathbf{R}_{a_i}<\dim\mathbf{R}$.

*General case:* $\mathbf{R}$ *arbitrary.* Apply the General Constructive Rereading Principle. Precisely this gives the following computation. First we remark that since $f$ is primitive, say $f=\sum_{j=0}^m a_jX^j$, the multiplicative subsets $M_m=\mathcal{M}(a_m)$, $M_{m-1}=\mathcal{S}_{\mathbf{R}}(a_m;a_{m-1})$, $\ldots$, $M_k=\mathcal{S}_{\mathbf{R}}(a_m,\ldots,a_{k+1};a_k)$, $\ldots$, $M_0=\mathcal{S}_{\mathbf{R}}(a_m,\ldots,a_1;a_0)$ are comaximal in $\mathbf{R}$. It is now sufficient to prove the conclusion for each ring $\mathbf{R}_{M_i}$. And this conclusion is obtained from the proof given for the first case. □

**Remark 1.6** If $\mathbf{R}$ is a valuation domain then any $f \in \mathbf{R}[X]$ is easily written as $f = ag$ where $a \in \mathbf{R}$ and $g \in \mathbf{R}[X]$ is primitive, invertible in $\mathbf{R}(X)$. From this fact, it follows easily that $\mathbf{R}(X)$ is again a valuation domain, and if $\mathrm{Kdim}\,\mathbf{R} \le d$ then $\mathrm{Kdim}\,\mathbf{R}(X) \le d$. So by Theorem 1.5, we painlessly get constructively that:

(i) If $\mathbf{R}$ is a valuation domain with $\mathrm{Kdim}\,\mathbf{R} \le 1$ then $\mathbf{R}\langle X\rangle$ is a Prüfer domain with $\mathrm{Kdim}\,\mathbf{R} \le 1$. As a matter of fact, it is clear that in this case, in one of the $\mathbf{R}\langle X\rangle_{\mathcal{M}_i}$, the computations are done like in $\mathbf{R}(X)$, while the other $\mathbf{R}\langle X\rangle_{\mathcal{M}_i}$ are localizations of the polynomial ring $\mathbf{K}[X]$ where $\mathbf{K}$ is the quotient field of $\mathbf{R}$.

(ii) If $\mathbf{R}$ is a Prüfer domain with $\mathrm{Kdim}\,\mathbf{R} \le 1$ then so is $\mathbf{R}\langle X\rangle$.
This is obtained from (i) by application of the General Constructive Rereading Principle.

**Remark 1.7** If $\mathrm{Kdim}\,\mathbf{R} = 0$ then clearly $\mathbf{R}\langle X\rangle = \mathbf{R}(X)$ (the rings $\mathbf{R}_{a_i}$ in Theorem 1.5 being trivial).

# 2 A constructive proof of the Quillen-Suslin theorem

We recall here the main steps of the constructive proof obtained in [1, 23] by deciphering Quillen's proof of the Quillen-Suslin theorem (a slightly more involved constructive deciphering was first given in [21]).

## 2.1 The patchings of Quillen and Vaserstein

We will state the following theorem without proof. Constructive proofs can be found in [1, 23].

**Theorem 2.1** (Vaserstein's patching, constructive form)
*Let $M$ be a matrix in $\mathbf{R}[X]$ and consider $S_1, \dots, S_n$ comaximal multiplicative subsets of $\mathbf{R}$. Then $M(X)$ and $M(0)$ are equivalent over $\mathbf{R}[X]$ if and only if, for each $1 \le i \le n$, they are equivalent over $\mathbf{R}_{S_i}[X]$.*

**Theorem 2.2** (Quillen's patching, constructive form)
*Let $P$ be a finitely presented module over $\mathbf{R}[X]$ and consider $S_1, \dots, S_n$ comaximal multiplicative subsets of $\mathbf{R}$. Then $P$ is extended from $\mathbf{R}$ if and only if for each $1 \le i \le n$, $P_{S_i}$ is extended from $\mathbf{R}_{S_i}$.*

**Proof** This is a corollary of the previous theorem since the isomorphism between $P(X)$ and $P(0)$ is nothing but the equivalence of two matrices $A(X)$ and $A(0)$ constructed from a relation matrix $M \in \mathbf{R}^{q\times m}$ of $P \simeq \mathrm{Coker}\,M$ (see [11]):

$$A(X) = \begin{pmatrix} M(X) & 0_{q,q} & 0_{q,q} & 0_{q,m} \\ 0_{q,m} & \mathrm{I}_q & 0_{q,q} & 0_{q,m} \end{pmatrix}.$$

□

## 2.2 Horrocks' theorem

Local Horrocks' theorem is the following result. A constructive proof can be found in [1, 23]. In fact the proof in [11] is quasi constructive.

**Theorem 2.3** (Local Horrocks extension theorem)
*If* $\mathbf{R}$ *is a residually discrete local ring and* $P$ *a finitely generated projective module over* $\mathbf{R}[X]$ *which is free over* $\mathbf{R}\langle X\rangle$*, then it is free over* $\mathbf{R}[X]$ *(i.e., extended from* $\mathbf{R}$*).*

A global version is obtained from a constructive proof of the local one by the Quillen's patching and applying the General Constructive Rereading Principle.

**Theorem 2.4** (Global Horrocks extension theorem)
*Let* $S$ *be the multiplicative set of monics in* $\mathbf{R}[X]$*,* $\mathbf{R}$ *an arbitrary commutative ring. If* $P$ *is a finitely generated projective module over* $\mathbf{R}[X]$ *such that* $P_S$ *is extended from* $\mathbf{R}$*, then* $P$ *is extended from* $\mathbf{R}$*.*

## 2.3 Quillen Induction

Classical Quillen Induction is the following one.

**Quillen Induction** *Suppose that a class of rings* $\mathcal{P}$ *satisfies the following properties:*

(i) *If* $\mathbf{R} \in \mathcal{P}$ *then* $\mathbf{R}\langle X\rangle \in \mathcal{P}$*.*

(ii) *If* $\mathbf{R} \in \mathcal{P}$ *then* $\mathbf{R}_{\mathfrak{m}} \in \mathcal{P}$ *for any maximal ideal* $\mathfrak{m}$ *of* $\mathbf{R}$*.*

(iii) *If* $\mathbf{R} \in \mathcal{P}$ *and* $\mathbf{R}$ *is local, and if* $M$ *is a finitely generated projective* $\mathbf{R}[X]$*-module, then* $M$ *is extended from* $\mathbf{R}$ *(that is, free).*

*Then, for each* $\mathbf{R} \in \mathcal{P}$*, if* $M$ *is a finitely generated projective* $\mathbf{R}[X_1, \ldots, X_n]$*-module, then* $M$ *is extended from* $\mathbf{R}$*.*

Quillen Induction needs maximal ideals, it works in classical mathematics but it cannot be fully constructive. The fact that (ii) and (iii) imply the case $n = 1$ in the conclusion needs a priori a constructive rereading, where one replaces Quillen's patching with maximal ideals by the constructive form (Theorem 2.2) with comaximal multiplicative subsets.

On the contrary, the "inductive step" in the proof is elementary (see e.g., [12]) and is based only on the following hypotheses.

(i) If $\mathbf{R} \in \mathcal{P}$ then $\mathbf{R}\langle X\rangle \in \mathcal{P}$.

(iii') If $\mathbf{R} \in \mathcal{P}$ and $M$ is a finitely generated projective $\mathbf{R}[X]$-module, then $M$ is extended from $\mathbf{R}$.

In the case of Serre'problem, $\mathbf{R}$ is a discrete field. So (i) and (iii') are well known. Remark that (iii') is also given by Horrocks' global theorem. So Quillen's proof is deciphered in a fully constructive way. Moreover, since a zero-dimensional reduced local ring is a discrete field we obtain the well known following generalization (see [2]).

**Theorem 2.5** (Quillen-Suslin, non-Noetherian version)

1. *If* $\mathbf{R}$ *is a zero-dimensional reduced ring then any finitely generated projective module over* $\mathbf{R}[X_1, \ldots, X_n]$ *is extended from* $\mathbf{R}$ *(i.e., isomorphic to a direct sum of modules* $\langle e_i\rangle$ *where the* $e_i$*'s are idempotent elements of* $\mathbf{R}$*).*

2. *As a particular case, any finitely generated projective module of constant rank over* $\mathbf{R}[X_1,\dots,X_n]$ *is free.*
3. *More generally the results work for any zero-dimensional ring.*

**Proof** The first point can be obtained from the local case by the constructive Quillen's patching. It can also be viewed as a concrete application of the General Constructive Rereading Principle. Let us denote by $\mathbf{R}_{red}$ the reduced ring associated to a ring $\mathbf{R}$. The third point follows from the fact that the canonical map $M \mapsto M_{red}$, $\mathrm{GK}_0(\mathbf{R}) \to \mathrm{GK}_0(\mathbf{R}_{red})$ is a bijection ($\mathrm{GK}_0(\mathbf{R})$ is the set isomorphism classes of finitely generated projective $\mathbf{R}$-modules). Moreover $\mathbf{R}_{red}[X_1,\dots,X_n] = \mathbf{R}[X_1,\dots,X_n]_{red}$. □

## 3 The Lequain-Simis Induction Theorem

In order to generalize the Quillen-Suslin theorem to Prüfer domains and seeing that the class of Prüfer domains is not stable under the formation $\mathbf{R}\langle X\rangle$, Lequain and Simis [13] found a clever way to bypass this difficulty by proving the following new induction theorem.

**Lequain-Simis Induction** *Suppose that a class of rings* $\mathcal{F}$ *satisfies the following properties:*

(i) *If* $\mathbf{R}\in\mathcal{F}$*, then every nonmaximal prime ideal of* $\mathbf{R}$ *has finite height.*

(ii) $\mathbf{R}\in\mathcal{F} \Rightarrow \mathbf{R}[X]_{\mathfrak{p}[X]}\in\mathcal{F}$ *for any prime ideal* $\mathfrak{p}$ *of* $\mathbf{R}$.

(iii) $\mathbf{R}\in\mathcal{F} \Rightarrow \mathbf{R}_{\mathfrak{p}}\in\mathcal{F}$ *for any prime ideal* $\mathfrak{p}$ *of* $\mathbf{R}$.

(iv) $\mathbf{R}\in\mathcal{F}$ *and* $\mathbf{R}$ *local* $\Rightarrow$ *any finitely generated projective module over* $\mathbf{R}[X]$ *is free.*

*Then, for each* $\mathbf{R}\in\mathcal{F}$*, if* $M$ *is a finitely generated projective* $\mathbf{R}[X_1,\dots,X_n]$*-module, then* $M$ *is extended from* $\mathbf{R}$.

Note here that if $\mathbf{R}$ is local with maximal ideal $\mathfrak{m}$, then $\mathbf{R}(X)=\mathbf{R}[X]_{\mathfrak{m}[X]}$.

We propose here a constructive variation of Lequain-Simis Induction Theorem using a simple proof. This is one important application of our dynamical comparison between the rings $\mathbf{R}(X)$ and $\mathbf{R}\langle X\rangle$.

**Theorem 3.1** (Constructive induction theorem) *Let* $\mathcal{F}$ *be a class of commutative rings with finite Krull dimensions satisfying the properties below:*

(ii') *If* $\mathbf{R}\in\mathcal{F}$ *then* $\mathbf{R}(X)\in\mathcal{F}$.

(iii) $\mathbf{R}\in\mathcal{F} \Rightarrow \mathbf{R}_S\in\mathcal{F}$ *for each monoid* $S$ *in* $\mathbf{R}$.

(iv') *If* $\mathbf{R}\in\mathcal{F}$ *then any finitely generated projective module over* $\mathbf{R}[X]$ *is extended from* $\mathbf{R}$.

*Then, for each* $\mathbf{R}\in\mathcal{F}$*, if* $M$ *is a finitely generated projective* $\mathbf{R}[X_1,\dots,X_n]$*-module, then* $M$ *is extended from* $\mathbf{R}$.

**Proof** We reason by double induction on the number $n$ of variables and the Krull dimension of the basic ring $\mathbf{R}$.
For the initialization of the induction there is no problem since if $n=1$ there is nothing to prove and for polynomial rings over zero-dimensional rings (see Theorem 2.5) the result is true constructively.
We assume that the construction is given with $n$ variables for rings in $\mathcal{F}$. Then we consider the case of $n+1$ variables and we give the proof by induction on the dimension of the ring $\mathbf{R}\in\mathcal{F}$.

We assume that the dimension is $\leq d+1$ with $d \geq 0$ and the construction has been done for rings of dimension $\leq d$.
Let $P$ be a finitely generated projective $\mathbf{R}[X_1,\dots,X_n,Y]$-module. Let us denote $X$ for $X_1,\dots,X_n$. The module $P$ can be seen as the cokernel of a presentation matrix $M = M(X,Y)$ with entries in $\mathbf{R}[X,Y]$. Let $A(X,Y)$ be the associated enlarged matrix (as in the proof of Theorem 2.2).
Using the induction hypothesis over $n$ and (ii') we know that $A(X,Y)$ and $A(0,Y)$ are equivalent over the ring $\mathbf{R}(Y)[X]$. This means that there exist matrices $Q_1, R_1$ with entries in $\mathbf{R}[X,Y]$ such that

$$Q_1 A(X,Y) \quad = \quad A(0,Y) R_1 \tag{3}$$
$$\text{where} \quad \det(Q_1) \quad \text{and} \quad \det(R_1) \qquad \text{are primitive polynomials in } \mathbf{R}[Y]. \tag{4}$$

We first want to show that $A(X,Y)$ and $A(0,Y)$ are equivalent over $\mathbf{R}\langle Y\rangle[X]$. Using the Vaserstein's patching, for doing this job it is sufficient to show that $A$ and $A(0,Y)$ are equivalent over $\mathbf{R}\langle Y\rangle[X]_{\mathcal{M}_i}$ for comaximal multiplicative subsets $\mathcal{M}_i$.
We consider the primitive polynomial $f = \det(Q_1)\det(R_1) \in \mathbf{R}[Y]$ and we apply Theorem 1.5. We get comaximal subsets $\mathcal{M}_1,\dots,\mathcal{M}_s$ of $\mathbf{R}\langle Y\rangle$ such that for each $1 \leq i \leq s$, either $f$ is invertible in $\mathbf{R}\langle Y\rangle_{\mathcal{M}_i}$ or $\mathbf{R}\langle Y\rangle_{\mathcal{M}_i}$ is a localization of $\mathbf{R}_{S_i}[Y]$ for some monoid $S_i \subseteq \mathbf{R}$ such that $\mathbf{R}_{S_i}$ has Krull dimension $\leq d$.
In the first case $\det(Q_1)$ and $\det(R_1)$ are invertible in $\mathbf{R}\langle Y\rangle_{\mathcal{M}_i}$. This implies that $A(X,Y)$ and $A(0,Y)$ are equivalent over $\mathbf{R}\langle Y\rangle[X]_{\mathcal{M}_i}$.
In the second case, by induction hypothesis over the dimension, $A(X,Y)$ and $A(0,0)$ are equivalent over $\mathbf{R}_{S_i}[Y][X]$. An immediate consequence is that $A(X,Y)$ and $A(0,Y)$ are equivalent over $\mathbf{R}_{S_i}[Y][X]$. Finally they are also equivalent over $\mathbf{R}\langle Y\rangle[X]_{\mathcal{M}_i}$ which is a localization of the previous ring.
Now we know that there exist invertible matrices $Q, R$ over the ring $\mathbf{R}\langle Y\rangle[X] \subseteq (\mathbf{R}[X])\langle Y\rangle$ such that

$$Q\,A(X,Y) = A(0,Y)\,R.$$

We know also that $A(0,0)$ and $A(0,Y)$ are equivalent over $\mathbf{R}[Y] \subseteq (\mathbf{R}[X])\langle Y\rangle$ (case $n=1$) and $A(0,0)$ and $A(X,0)$ are equivalent over $\mathbf{R}[X] \subseteq (\mathbf{R}[X])\langle Y\rangle$. So $A(X,0)$ and $A(X,Y)$ are equivalent over $(\mathbf{R}[X])\langle Y\rangle$, and by virtue of global Horrocks' theorem (Theorem 2.4), $P$ is extended from $\mathbf{R}[X]$, i.e., $A(X,0)$ and $A(X,Y)$ are equivalent over $\mathbf{R}[X,Y]$. By induction hypothesis, $P$ is extended from $\mathbf{R}$. □

Recall that a ring is called a pp-ring if the annihilator ideal of any element is generated by an idempotent.

**Corollary 3.2** (Lequain-Simis Theorem) *For any finite-dimensional arithmetical pp-ring* $\mathbf{R}$*, all finitely generated projective* $\mathbf{R}[X_1,\dots,X_n]$*-modules,* $n \geq 2$*, are extended from* $\mathbf{R}$ *if and only if all finitely generated projective* $\mathbf{R}[X_1]$*-modules are extended from* $\mathbf{R}$*.*

**Proof** We prove that the class $\mathcal{F}$ of finite-dimensional arithmetical pp-rings such that all finitely generated projective $\mathbf{R}[X_1]$-modules are extended from $\mathbf{R}$ satisfies the hypothesis in our induction theorem. Only the first point (ii') is problematic. We assume to have a constructive proof in the local case, i.e., the case of valuation domains. So, starting with an arithmetical pp-ring, the General Constructive Rereading Principle gives comaximal multiplicative sets where the needed computations are done successfully. This allows to give the desired global conclusion in an explicit way. □

**Remark 3.3** Thierry Coquand announced recently a constructive proof of the Bass-Simis-Vasconcelos theorem (projective modules over $\mathbf{V}[X]$, $\mathbf{V}$ a valuation domain, are free).

As always constructive proofs work in classical mathematics and Theorem 3.1 applies. Moreover, in classical mathematics, we get the following variation:

**Theorem 3.4** (New classical induction theorem) *Let $\mathcal{F}$ be a class of commutative rings with finite Krull dimensions satisfying the properties below:*

(ii) *If $\mathbf{R} \in \mathcal{F}$ and $\mathbf{R}$ is local then $\mathbf{R}(X) \in \mathcal{F}$.*

(iii') $\mathbf{R} \in \mathcal{F} \Rightarrow \mathbf{R}_S \in \mathcal{F}$ *for each multiplicative set $S$ in $\mathbf{R}$.*

(iv) *If $\mathbf{R} \in \mathcal{F}$ and $\mathbf{R}$ is local then any finitely generated projective module over $\mathbf{R}[X]$ is extended from $\mathbf{R}$.*

*Then, for each $\mathbf{R} \in \mathcal{F}$, if $M$ is a finitely generated projective $\mathbf{R}[X_1, \ldots, X_n]$-module, then $M$ is extended from $\mathbf{R}$.*

**Proof** From (ii) and (iv) we deduce (ii') and (iv') in Theorem 3.1 by using the abstract Quillen's patching that uses maximal ideals. □

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