GENERATING NON-NOETHERIAN MODULES CONSTRUCTIVELY

THIERRY COQUAND, HENRI LOMBARDI, CLAUDE QUITTÉ

Abstract. In [6], Heitmann gives a proof of a Basic Element Theorem, which has as corollaries some versions of the "Splitting-off" theorem of Serre and the Forster-Swan theorem in a non Noetherian setting. We give elementary and constructive proofs of such results. We introduce also a new notion of dimension for rings, which is only implicit in [6] and we present a generalisation of the Forster-Swan theorem, answering a question left open in [6].

1. Zariski spectrum and Krull dimension

Let R be a commutative ring with unit. Following Joyal [7], we define the Zariski spectrum of R as the distributive lattice generated by symbols $D(f),\ f \in R$ and relations

$$D(0) = 0 \qquad D(1) = 1 \quad D(fg) = D(f) \wedge D(g) \qquad D(f+g) \leq D(f) \vee D(g)$$

We write $D(f_1, \ldots, f_m)$ for $D(f_1) \vee \cdots \vee D(f_m)$. For $m = 0$ we have $D() = 0$. It can be shown directly that

$$D(g_1) \wedge \cdots \wedge D(g_n) \leq D(f_1, \ldots, f_m)$$

holds if and only if the monoid generated by $g_1, \ldots, g_n$ meets the ideal generated by $f_1, \ldots, f_m$ [1]. Thus $D(f_1, \ldots, f_m)$ can be defined as the radical of the ideal generated by $f_1, \ldots, f_m$ (with inclusion as ordering), and we have a point-free and elementary description of the basic open sets of the Zariski spectrum of R.

In [2] we present the following elementary characterization of Krull dimension. If $a \in R$ we define the *boundary* of a as being the the ideal N_a generated by a and the elements b such that ab is nilpotent (or equivalently $D(ab) = 0$).

Theorem 1.1. *The dimension of R is $<n+1$ if and only if for all $a \in R$ the dimension of R/N_a is $<n$.*

This can actually be taken as a constructive definition of Krull dimension, if we define a ring R to be of dimension <0 if and only if R is trivial. This inductive definition of being of dimension $<n$ is then equivalent to the usual definition that there is no strictly increasing chain of prime ideals of length n [2]. In [1] it is shown, in an elementary and constructive way, that the dimension of a polynomial ring with n variables over a field is $\leq n$.

2. The stable range theorem

All the arguments will be based on the following trivial remark, that we state explicitely since it will motivate the notion of dimension that we present in section 4.

Lemma 2.1. *If b is nilpotent then $1 = D(b_1, \ldots, b_k) \vee D(b)$ implies $1 = D(b_1, \ldots, b_k)$. More generally, if $b \in R$ is nilpotent in $R[a^{-1}]$ then $D(a) \leq D(b_1, \ldots, b_k) \vee D(b)$ implies $D(a) \leq D(b_1, \ldots, b_k)$.*

Proof. Indeed, b is nilpotent if and only if $D(b) = 0$. Likewise, $b \in R$ is nilpotent in $R[a^{-1}]$ if and only if $D(ab) = 0$, and $D(a) \leq D(b_1, \ldots, b_k) \vee D(b)$ implies $D(a) \leq D(b_1, \ldots, b_k) \vee D(ab)$. □

We shall need also the following two remarks.

Lemma 2.2. $D(y + b) \vee D(yb) = D(y) \vee D(b)$

Lemma 2.3. *If by is nilpotent then $1 = D(b_1, \ldots, b_k, b, y)$ implies $1 = D(b_1, \ldots, b_k, b + y)$. More generally, if $by \in R$ is nilpotent in $R[a^{-1}]$ then $D(a) \leq D(b_1, \ldots, b_k, b, y)$ implies $D(a) \leq D(b_1, \ldots, b_k, b + y)$.*

Proof. By Lemmas 2.1 and 2.2. □

Notice that the nilpotent hypothesis was used only to invoke lemma 2.1.

Our inductive definition of dimension allows more perspicuous proofs. For instance, as a motivation of our method, here is a proof of the "Stable Range" theorem.

Theorem 2.4. *If the dimension of R is $< n$ and $1 = D(a, b_1, \ldots, b_n)$ there exists $x_1, \ldots, x_n$ such that $1 = D(b_1 + ax_1, \ldots, b_n + ax_n)$.*

Proof. The proof is by induction on n. This is clear if $n = 0$, since in this case the ring is trivial and we have $1 = D()$. If $n > 0$, let I be the ideal boundary of b_n. We have $b_n \in I$ and the dimension of R/I is $<n - 1$. By induction, we can find $x_1, \ldots, x_{n-1}$ such that

$$1 = D(b_1 + ax_1, \ldots, b_{n-1} + ax_{n-1})$$

in R/I. This means that there exists x_n such that $D(b_n x_n) = 0$ and

$$1 = D(b_1 + ax_1, \ldots, b_{n-1} + ax_{n-1}) \vee D(b_n) \vee D(x_n)$$

Since

$$1 = D(b_1 + ax_1, \ldots, b_{n-1} + ax_{n-1}) \vee D(b_n) \vee D(a)$$

this implies by distributivity

$$1 = D(b_1 + ax_1, \ldots, b_{n-1} + ax_{n-1}) \vee D(b_n) \vee D(ax_n)$$

hence the result by Lemma 2.3. □

It follows then for instance directly that a stably free module of rank $\geq n$ over a ring of dimension $<n$ is free [9], without any noetherian hypotheses. We can in the same way prove Kronecker's theorem about algebraic sets [3, 8].

We shall need a variation on this result. If $L \in R^n$ is a vector $(a_1, \ldots, a_n)$ we write $D(L)$ for $D(a_1, \ldots, a_n)$.

Lemma 2.5. *If $a \in R$ and the dimension of $R[a^{-1}]$ is $<n$ then for any $L \in R^n$ there exists $X \in R^n$ such that $D(a) \leq D(L - aX)$. Furthermore, we can find X of the form aY, $Y \in R^n$.*

Proof. We let L be $(b_1, \dots, b_n)$ and we reason by induction on n. This is clear if $n = 0$. If $n > 0$ let N be the ideal boundary of b_n *in* $R[a^{-1}]$, and I the ideal $N \cap R$. It can be checked that I that $(R/I)[a^{-1}]$ is isomorphic to $R[a^{-1}]/N$. Hence we can apply the induction hypothesis to R/I and compute $(x_1, \dots, x_{n-1}) \in R^{n-1}$ such that

$$D(a) \leq D(b_1 - ax_1, \dots, b_{n-1} - ax_{n-1})$$

in R/I. In turn, this means that we can find x_n such that $D(ax_nb_n) = 0$ and

$$D(a) \leq D(b_1 - ax_1, \dots, b_{n-1} - ax_{n-1}) \vee D(b_n) \vee D(x_n)$$

in R. This implies

$$D(a) \leq D(b_1 - ax_1, \dots, b_{n-1} - ax_{n-1}) \vee D(b_n) \vee D(ax_n)$$

hence the result by Lemma 2.3. Finally, as we can apply the result with a^2 instead of a since $R[a^{-2}] = R[a^{-1}]$ and $D(a) = D(a^2)$, we get the result with $X = aY$. □

Corollary 2.6. *Let M be a $n \times n$ matrix of element in R and δ its determinant. If the dimension of $R[\delta^{-1}]$ is $<n$ then for each $C \in R^n$ there exists $X \in R^n$ such that $D(\delta) \leq D(MX - C)$. Furthermore we can find X of the form δY, $Y \in R^n$.*

Proof. The proof is based on Cramer formulae. Let $\widetilde{M}$ be the adjoint matrix of M, and $L = \widetilde{M}C$. We have then $\widetilde{M}(MX - C) = \delta X - L$ for an arbitrary column vector $X \in R^n$. Hence the ideal generated by the coordinates of $\delta X - L$ is included in the one generated by the coordinates of $MX - C$, and

$$D(\delta X - L) \leq D(MX - C)$$

By Lemma 2.5 we can find *one* $X = \delta Y \in R^n$ such that $D(\delta) \leq D(\delta X - L)$, and hence $D(\delta) \leq D(MX - C)$ as desired. □

3. A Basic Element Theorem

Let F be a rectangular matrix of elements in R of columns $C_0, C_1, \dots, C_p$, and G the matrix of columns $C_1, C_2, \dots, C_p$. Let Δ_n be $\bigvee_\nu D(\nu)$ where ν varies over all minors of F of order n.

Theorem 3.1. *Fix $0<n \leq p$. Suppose that for each minor ν of G of order n the ring $R[\nu^{-1}]$ is of dimension $<n$. Then there exist $t_1, \dots, t_p$ such that $\Delta_n \leq D(C_0 + t_1C_1 + \cdots + t_pC_p)$ and $D(C_0) \leq D(C_0 + t_1C_1 + \cdots + t_pC_p)$.*

Proof. Let Δ'_n be $\bigvee_\nu D(\nu)$ where ν varies over all minors of G of order n. For any $t_1, \dots, t_p$ and any minor ν of F which uses the column C_0 we have $D(\nu) \leq \Delta'_n \vee D(C_0 + t_1C_1 + \cdots + t_pC_p)$ and so it suffices to prove the theorem with Δ_n replaced by Δ'_n.

It is also enough to show that for *one* minor ν of G of order n we can find $t_1, \dots, t_p$ such that $D(\nu) \leq D(C_0 + t_1C_1 + \cdots + t_pC_p)$ and $D(C_0) \leq D(C_0 + t_1C_1 + \cdots + t_pC_p)$ because we can then apply this successively to all minors of G of order n. But this is a direct consequence of Corollary 2.6: we find $t_1, \ldots, t_p$ (with $t_i = 0$ for the columns outside the minor ν) that are multiple of ν and such that

$$D(\nu) \leq D(C_0 + t_1 C_1 + \cdots + t_p C_p)$$

Since $t_1, \ldots, t_p$ are all multiple of ν we have also $D(C_0) \leq D(\nu) \vee D(C_0 + t_1 C_1 + \cdots + t_p C_p)$ and hence $D(C_0) \leq D(C_0 + t_1 C_1 + \cdots + t_p C_p)$ as required. □

Corollary 3.2. *Suppose that* $1 = \Delta_1$ *and that for each* $k > 0$ *and for each minor* ν *of* G *of order* n *the ring* $R[\nu^{-1}]/\Delta_{n+1}$ *is of dimension* $<n$. *Then there exist* $t_1, \ldots, t_p$ *such that the vector* $C_0 + t_1 C_1 + \cdots + t_p C_p$ *is unimodular.*

Proof. In the statement of this corollary, we identify Δ_l with its corresponding radical ideal. Using Theorem 3.1, we define a sequence of vectors C_0^i, $i = 1, \ldots$ with $C_0^1 = C_0$. For each $k > 0$, reasoning in R/Δ_{k+1}, we build C_0^{k+1} of the form $C_0^k + u_1 C_1 + \cdots + u_p C_p$ such that $\Delta_k \leq D(C_0^{k+1})$ and $D(C_0^k) \leq D(C_0^{k+1})$ in R/Δ_{k+1}. This means that we have, in R

$$D(C_0^k) \vee \Delta_k \leq D(C_0^{k+1}) \vee \Delta_{k+1}$$

Hence the result since $\Delta_1 = 1$ and $\Delta_k = 0$ for n large enough. □

From this follows directly, as in [4, 6], a version of Serre's "Splitting-off" theorem and the Forster-Swan theorem with Krull dimension and without noetherian hypothesis. For instance, here is a version of Forster's theorem [5].

Corollary 3.3. *Let* M *be a module which is finitely generated over a ring* R *of dimension* $\leq d$. *If* M *is locally generated by* r *elements then* M *can be generated by* $d + r$ *elements.*

Proof. M is a quotient of a finitely presented module M' which has a Fitting ideal of order r which contains 1, and we can as well suppose that $M' = M$. Let $m_0, m_1, \ldots, m_p$ be a system of generators of M and F be a presentation matrix of M. If $p \geq d + r$ we have $1 = \Delta_{d+1}(F)$ and using theorem 3.1, we can find $t_1, \ldots, t_p$ such that M is generated by $m_1 - t_1 m_0, \ldots, m_p - t_p m_0$. Hence we can generate M by p elements. □

Using Corollary 3.2, one could give a more sophisticated version of this result, as in [6]. The next section shows how to recover an improved version of these theorems with Heitmann's notion of j-spectrum [6], also without noetherian hypothesis.

4. A New Notion of Dimension

By analogy with our inductive definition of Krull dimension, we define now when a ring R is of H-dimension $<n$. Let J be the Jacobson radical of R, that is the ideal of elements x such that $1 - xy$ is invertible for all $y \in R$. We redefine N_a, ideal boundary of a, as the ideal generated by a and the elements b such that $ab \in J$. Thus we replace the radical of R, intersection of all prime ideals, by its Jacobson radical, intersection of all maximal ideals. A ring of H-dimension <0 is a trivial ring, and R is of H-dimension $<n+1$ if and only if for any $a \in R$ the H-dimension of R/N_a is $<n$. The proofs of all our previous results go through directly with this new definition of dimension. This follows from the following

elementary result, which shows that Lemma 2.1 holds with our new notion, and also that this new notion is in some sense optimal.

Lemma 4.1. *$b \in J$ if and only if for all $b_1, \ldots, b_k$, we have $D(b_1, \ldots, b_k) = 1$ whenever $D(b_1, \ldots, b_k) \vee D(b) = 1$. More generally $b \in R$ is in the Jacobson radical of $R[a^{-1}]$ if and only if $D(a) \leq D(b_1, \ldots, b_k)$ whenever $D(a) \leq D(b_1, \ldots, b_k) \vee D(b)$.*

Proof. Assume that $D(b_1, b) = 1$ implies $D(b_1) = 1$. Since $1 - by + by = 1$ we get that $1-by$ is invertible for all y, that is $b \in J$. Conversely if $b \in J$ and $y_1b_1 + \cdots + y_kb_k + by = 1$ then, since $1 - by$ is invertible, we have $1 = D(b_1, \ldots, b_k)$. The proof of the second part is similar. □

We get in this way a version of the Basic Element Theorem 3.1 with H-dimension instead.

Theorem 4.2. *Fix $n \leq p$. Suppose that for each minor ν of G of order n the ring $R[\nu^{-1}]$ is of H-dimension $<n$. Then there exist $t_1, \ldots, t_p$ such that $\Delta_n \leq D(C_0 + t_1C_1 + \cdots + t_pC_p)$* and *$D(C_0) \leq D(C_0 + t_1C_1 + \cdots + t_pC_p)$.*

Corollary 4.3. *Suppose that for each $n \leq p$ and for each minor ν of G of order n the ring $R[\nu^{-1}]/\Delta_{n+1}$ is of H-dimension $<n$. Then there exist $t_1, \ldots, t_p$ such that the vector $C_0 + t_1C_1 + \cdots + t_pC_p$ is unimodular.*

Our notion of H-dimension is only implicit in [6]. Heitmann introduces instead the j-spectrum of R which is the closure of the maximal spectrum of R in the patch topology, with the topology induced by Zariski topology, and defines the j-dimension of R to be the (Krull) dimension of the j-Spec(R). Recall that Heitmann's j-spectrum is not the usual one given in the literature. This new spectrum was introduced in [6] in order to deal with the non Noetherian case.

Proposition 4.4. *If the j-dimension of R is $\leq n$ then its H-dimension is also $\leq n$.*

Proof. By induction on n. Let X be j-Spec R. For $a \in R$, the boundary ideal N_a corresponds to a closed subset $Y = V(a) \cap \overline{D(a) \cap X} = \mathrm{Spec}(R/N_a)$ (as in [6], $^-$indicates closure in the Zariski topology). The subset Y being closed, its closed (=maximal) points are exactly the points in Max(R) $\cap Y$. This implies j-Spec$(R/N_a) \subseteq Y \cap X$. It is then enough to show that the (Krull) dimension of $Y \cap X$ is $<n$. But we have $Y \cap X \subseteq \overline{V(a) \cap X} \cap \overline{D(a) \cap X} \cap X$ which has dimension $<n$ by Lemma 1.2 in [6] (this is the boundary of $D(a) \cap X$ in X). □

This means that the H-dimension is $\leq$ the j-dimension. We conjecture that the two notions of dimension, H-dimension and j-dimension, may differ in general. So our results could improve those of [6] in some cases. Nevertheless we think that the main importance of our framework is the constructive and elementary character of our proofs. The next section illustrates this point, by answering a question left open at the end of [6].

5. A generalisation of the Forster-Swan theorem

The following result is a refinement of Theorem 2.4.

Lemma 5.1. *Assume $L, L_1 \ldots, L_n$ are vectors of the same length. If H-dim(R)<n and*

$$1 = D(a, b_1, \ldots, b_n) \vee D(L)$$

then there exist $x_1, \ldots, x_n$, all multiple of a, such that

$$1 = D(b_1 + ax_1, \ldots, b_n + ax_n) \vee D(L + x_1 L_1 + \cdots + x_n L_n)$$

Proof. The proof is by induction on n. This is clear if $n = 0$. If $n > 0$, let I be the ideal boundary of b_n. We have $b_n \in I$ and H-dim(R/I)<$n-1$. By induction, we can find $x_1, \ldots, x_{n-1}$ multiple of a such that

$$1 = D(b_1 + ax_1, \ldots, b_{n-1} + ax_{n-1}) \vee D(L + x_1 L_1 + \cdots + x_{n-1} L_{n-1})$$

in R/I. This means that there exists y such that $b_n y \in J$ and

$$1 = D(b_1 + ax_1, \ldots, b_{n-1} + ax_{n-1}) \vee D(L + x_1 L_1 + \cdots + x_{n-1} L_{n-1}) \vee D(b_n) \vee D(y)$$

in R. Take $x_n = ay$. Then x_n is a multiple of a and we claim

$$1 = D(b_1 + ax_1, \ldots, b_n + ax_n) \vee D(L + x_1 L_1 + \cdots + x_n L_n)$$

Indeed, if X is the right hand-side, we have

$$X \vee D(a) = D(a, b_1, \ldots, b_n) \vee D(L) = 1$$

since all $x_1, \ldots, x_n$ are multiple of a, but also, since x_n is a multiple of y

$$X \vee D(y) = D(b_1 + ax_1, \ldots, b_{n-1} + ax_{n-1}, b_n) \vee D(L + x_1 L_1 + \cdots + x_{n-1} L_{n-1}) \vee D(y)$$

and hence $X \vee D(y) = 1$. Also

$$X \vee D(b_n) = X \vee D(b_n) \vee D(ax_n) = 1$$

so that $X \vee D(b_n y) = 1$ and hence $X = 1$ since we can apply Lemma 4.1 to $b_n y \in J$. □

We can now prove a variation of Theorem 4.2.

Theorem 5.2. *Fix $n \leq p$. Suppose that R is of H-dimension <n and $\Delta_n = 1$. Then there exist $t_1, \ldots, t_p$ such that $1 = D(C_0 + t_1 C_1 + \cdots + t_p C_p)$.*

Let $\Delta_n(C_0; G)$ be $\bigvee_\nu D(\nu)$ where ν varies over all minors of F of order n using the column C_0.

Lemma 5.3. *If ν is a minor of G of order n using the columns $C_{i_1}, \ldots, C_{i_n}$ and*

$$\Delta_n(C_0; G) \vee D(\nu) = 1$$

and H-dim(R)< n then there exist $x_1, \ldots, x_n$ such that

$$\Delta_n(C_0 + x_1 C_{i_1} + \cdots + x_n C_{i_n}; G) = 1$$

Proof. This follows from lemma 5.1: we take a to be ν, b_k to be the minor obtained from ν by replacing C_{i_k} by C_0, L to be the vector of all remaining minors in $\Delta_n(C_0; G)$, and L_k is obtained by replacing C_0 by C_{i_k} in L. □

For proving Theorem 5.2, we write

$$\Delta_n = \Delta_n(C_0; G) \vee \bigvee_{\nu} D(\nu)$$

where ν varies over all minors of G of order n. Applying Lemma 5.3 to suitable quotients of R, we can eliminate successively these minors, replacing at each step C_0 by a vector of the form $C_0 + t_1C_1 + \cdots + t_pC_p$. At the end, we get $t_1, \ldots, t_p$ such that $\Delta_n(C_0 + t_1C_1 + \cdots + t_pC_p; G) = 1$ and this implies $D(C_0 + t_1C_1 + \cdots + t_pC_p) = 1$

We can now state the following result, which can be proved from Theorem 5.2 by arguments similar to the ones for Corollary 3.3.

Theorem 5.4. *If H-dim(R)$\leq d$ and if M is a finitely generated module over R which is locally generated by r elements, then M is generated by $d + r$ elements.*

Swan's theorem [10], which itself generalises Forster's theorem [5], can be seen as a special case when the maximal spectrum of R is Noetherian. This improves also on [6] that assumes instead j-dim$(R[a^{-1}])\leq d$ for all $a \in R$, and on [11] that obtained $r(d' + 1)$ generators with $d' = j$-dim(R).

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References

[1] Th. Coquand and H. Lombardi. Hidden constructions in abstract algebra (3) Krull dimension. in Commutative ring theory and applicatoins. Eds: Fontana, M, LNPAM 131.

[2] Th. Coquand, H. Lombardi and M.F. Roy. Une caractérisation élémentaire de la dimension de Krull. Preprint 2003.

[3] Th. Coquand. Sur un théorème de Kronecker concernant les variétés algébriques C. R. Acad. Sci. Paris, Ser. I 338 (2004), Pages 291-294

[4] D. Eisenbud and E. G. Evans, Jr. Generating modules efficiently: theorems from algebraic K-theory. J. Algebra 27 (1973), 278–305.

[5] O. Forster. Über die Anzahl der Erzeugenden eines Ideals in einem Noetherschen Ring. Math. Z. 84 1964 80–87.

[6] R. Heitmann. Generating non-Noetherian modules efficiently. Michigan Math. J. 31 (1984), no. 2, 167–180.

[7] A. Joyal. Le théorème de Chevalley-Tarski. *Cahiers de Topologie et Géométrie Différentielle* 16, 256–258 (1975).

[8] L. Kronecker. Grundzüge einer arithmetischen Theorie der algebraischen Grössen. *J. reine angew. Math.* 92, 1-123 (1882). Réimprimé dans *Leopold Kronecker's Werke*, II, 237–387.

[9] T.Y. Lam. Serre's conjecture. Lecture Notes in Mathematics, Vol. 635. Springer-Verlag, Berlin-New York, 1978.

[10] R.G. Swan. The Number of Generators of a Module. Math. Z. 102 (1967), 318-322.

[11] W. Vasconcelos, R. Wiegand. Bounding the number of generators of a module. Math. Z. 164 (1978), no. 1, 1–7.