An elementary characterisation of Krull dimension

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Abstract

We give an elementary characterisation of Krull dimension for distributive lattices and commutative rings. This follows the following geometrical intuition: an algebraic variety is of dimension $\leq k$ if and only if each subvariety has a boundary of dimension $< k$. Since our results hold for distributive lattices, they hold, by Stone duality [11], for any spectral spaces.

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Boundaries of an element in a distributive lattice

By *distributive lattice* we mean a lattice with a minimum and a maximum (so that all finite parts have a supremum and an infimum) which is distributive.

Let L be a distributive lattice. An *ideal* of L is a subset $I \subseteq L$ such that

$$\begin{array}{rcl} & & 0 \in I \\ x, y \in I & \Longrightarrow & x \vee y \in I \\ x \in I,\ z \in L & \Longrightarrow & x \wedge z \in I \end{array}$$

The last property can be written $(x \in I,\ y \leq x) \Rightarrow y \in I$.

The dual notion is the notion of *filter*. A filter F is a subset of L such that

$$\begin{array}{rcl} & & 1 \in F \\ x, y \in F & \Longrightarrow & x \wedge y \in F \\ x \in F,\ z \in L & \Longrightarrow & x \vee z \in F \end{array}$$

A *prime ideal* is an ideal I such that $1 \notin I$ and

$$x \wedge y \in I \Rightarrow [x \in I \text{ or } y \in I]$$

and dually a *prime filter* is a filter F such that $0 \notin F$ and

$$x \vee y \in F \Rightarrow [x \in F \text{ or } y \in F]$$

Notice that an ideal (resp. a filter) is prime if and only if its complement is a filter (resp. an ideal).

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If $x \in L$ we denote by $D(x)$ the set of prime ideals I such that $x \notin I$. We have $D(0) = \emptyset$ and $D(x) \cap D(y) = D(x \wedge y)$. The set of all prime ideals of a distributive lattice L has a natural structure of topological space, called the *spectrum* $Sp(L)$ of L. We take for basic open sets the sets $D(x),\ x \in L$. It can be shown that, each $D(x)$ is compact, and that the compact open sets of $Sp(L)$ are exactly the subsets of the form $D(x),\ x \in L$ [11].

The spaces (homeomorphic to spaces) of the form $Sp(L)$ are called *spectral spaces* and it is possible to characterise directly these spaces by topological properties [11, 7]. Most topological spaces used in commutative algebra, Zariski spectrum of a ring, spaces of valuations of a field, ..., are spectral spaces.

The set $Sp(L)$ is ordered by inclusion, and the *Krull dimension* of L is defined as the upper bound of the length of chains of prime ideals (or equivalently chains of prime filters).

If $x \in L$ we define the *boundary ideal* of x as being the ideal generated by x and the elements $y \in L$ such that $x \wedge y = 0$. Dually, we define the *boundary filter* of x as being the filter generated by x and the elements $y \in L$ such that $x \vee y = 1$.

Definition 1 *The upper boundary of $x \in L$ in the distributive lattice L is the distributive lattice $L^{\{x\}}$ quotient of L by the boundary ideal of x. Thus it is the lattice $L, \wedge, \vee$ with the order*

$$a \leq^x b \qquad \iff \qquad \exists y \in L \ \ (x \wedge y = 0 \ \ \& \ \ a \leq x \vee y \vee b)$$

When L is implicative the definition becomes $a \leq x \vee \neg x \vee b$.

By considering the dual lattice, one defines the lower boundary $L_{\{x\}}$, which is the distributive lattice quotient of L by the boundary filter of x. Thus it is the lattice $L, \wedge, \vee$ with the order

$$a \leq_x b \qquad \iff \qquad \exists y \in L \ \ (x \vee y = 1 \ \ \& \ \ a \wedge x \wedge y \leq b)$$

It can be checked that the boundary of the open $D(x)$, viewed as a subspace of $Sp(L)$, is a spectral space (as a closed set in a spectral space) and corresponds by Stone duality to the distributive lattice $L^{\{x\}}$.

Krull dimension of a distributive lattice

The duality between distributive lattice and spectral spaces relies on classical logic and the axiom of choice. From a constructive point of view, this duality is seen as a way to develop the theory of spectral spaces, using distributive lattices as a point-free presentation of these spaces [8]. One is thus led to look for direct definitions of topological notions in term of distributive lattices, and for instance, a direct definition of the Krull dimension.

A first constructive definition of Krull dimension was sketched in [1]. This definition was analysed in the work [6]. The author gave an elementary characterisation of the Krull dimension of a lattice L in term of the Boolean algebra generated by L. In [3], following the idea in [1], the two first authors proved the following result, which gives yet another a concrete characterization of Krull dimension.

Theorem 2 *Let L be a distributive lattice generated by a subset S and ℓ a nonnegative integer. The following are equivalent*

(1) *L has Krull dimension $\leq \ell$*

(2) *For all $x_0, \ldots, x_\ell \in S$ there exist $a_0, \ldots, a_\ell \in L$ such that*

$$a_0 \wedge x_0 \leq 0\,, \quad a_1 \wedge x_1 \leq a_0 \vee x_0\,, \ldots\,, \quad a_\ell \wedge x_\ell \leq a_{\ell-1} \vee x_{\ell-1}\,, \quad 1 \leq a_\ell \vee x_\ell.$$

In particular a distributive lattice L is of dimension ≤ 0 if and only if L is a Boolean algebra (any element has a complement).

The goal of this paper is to present a simpler inductive characterisation of Krull dimension, which provides also a simple proof of the equivalence between (1) and (2) in the previous theorem. This inductive characterisation corresponds to the following geometrical intuition: a variety is of dimension $\leq k$ if and only if any subvariety has a boundary of dimension $< k$. (the induction begins with dimension -1 which defines the trivial lattice).

Theorem 3 *Let L be a distributive lattice generated by a subset S and ℓ a nonnegative integer. The following are equivalent*

(1) *L has Krull dimension $\leq \ell$*

(2) *For all $x \in S$ the boundary $L^{\{x\}}$ is of Krull dimension $\leq \ell - 1$.*

(3) *For all $x \in S$ the boundary $L_{\{x\}}$ is of Krull dimension $\leq \ell - 1$.*

Proof.
(1) $\Leftrightarrow$ (2): We show first that any maximal filter F of L becomes trivial in $L^{\{x\}}$, i.e. it contains 0. This means that one can find $a \in F$ such that $a \leq^x 0$. If $x \in F$ this holds since $x \leq^x 0$. If $x \notin F$ there exists $z \in F$ such that $x \wedge z = 0$ (since the filter generated by F and x is trivial) and we have then $z \leq^x 0$. This shows that the Krull dimension of $L^{\{x\}}$ becomes one less than the one of L (if it is finite).
Next, we show that if $F' \subset F$, F maximal and $x \in F \setminus F'$ then F' does not become trivial in $L^{\{x\}}$ (which shows that *dim* $L^{\{x\}}$ is *dim* $L - 1$ with a good choice of x). Indeed, we would get otherwise $z \in F'$ such that $z \wedge x = 0$, which is impossible since both z and x are in F.
We finally notice that if $F' \subset F$ are distinct prime filters and S generates L one can find $x \in S$ such that $x \in F \setminus F'$.
(1) $\Leftrightarrow$ (3) is a consequence of (1) $\Leftrightarrow$ (2) by duality. □

By Stone duality [11], we get the following result.

Theorem 4 *A spectral space X is of Krull dimension $\leq k$ if and only if any open compact of X has a boundary of dimension $< k$.*

Since any spectral space can be viewed as the spectrum of a commutative ring, it is natural to define directly boundaries for commutative rings.

The two boundaries of an element in a commutative ring

Let R be a commutative ring. We write $\langle J\rangle$ for the ideal of R generated by the subset $J \subseteq R$. We write $\mathcal{M}(U)$ for the monoid (a monoid will always be multiplicative) generated by the subset $U \subseteq R$. Given a commutative ring R the *Zariski lattice* $\mathrm{Zar}(R)$ has for elements the radicals of finitely generated ideals. The order relation is the inclusion and we get

$$\sqrt{I} \wedge \sqrt{J} = \sqrt{IJ}, \quad \sqrt{I} \vee \sqrt{J} = \sqrt{I+J}.$$

We shall write $\widetilde{a}$ for $\sqrt{\langle a\rangle}$. We have

$$\widetilde{a_1} \vee \cdots \vee \widetilde{a_m} = \sqrt{\langle a_1, \ldots, a_m\rangle} \quad \text{and} \quad \widetilde{a_1} \wedge \cdots \wedge \widetilde{a_m} = \widetilde{a_1 \cdots a_m}.$$

Let U and J be two finite subsets of R, we have

$$\bigwedge_{u\in U} \widetilde{u} \leq_{\mathrm{Zar}(R)} \bigvee_{a\in J} \widetilde{a} \quad \Longleftrightarrow \quad \prod_{u\in U} u \in \sqrt{\langle J\rangle} \quad \Longleftrightarrow \quad \mathcal{M}(U) \cap \langle J\rangle \neq \emptyset$$

This describes completely the lattice $\mathrm{Zar}(R)$. More precisely ([3]) we have:

Proposition 5 *The lattice $\mathrm{Zar}(R)$ of a commutative ring R is (up to isomorphism) the lattice generated by symbols $D(x)$, $x \in R$ with the relations*

$$D(0) = 0, \quad D(1) = 1, \quad D(fg) = D(f) \wedge D(g), \quad D(f+g) \leq D(f) \vee D(g).$$

The spectrum of the distributive lattice $\operatorname{Zar}(R)$ is naturally isomorphic to the Zariski spectrum of the ring R. So the Krull dimension of a commutative ring R is the same as the Krull dimension of its Zariski lattice $\operatorname{Zar}(R)$.

Definition 6 *Let R be a commutative ring and $x \in R$.*

(1) *The boundary $R^{\{x\}}$ of x in R is the quotient ring $R/I^{\{x\}}$ where $I^{\{x\}} = xR + (\sqrt{0} : x)$.*

(2) *The boundary $R_{\{x\}}$ of x in R is the localized ring $R_{S_{\{x\}}}$ where $S_{\{x\}} = x^{\mathbb{N}}(1 + xR)$.*

The next proposition is easy.

Proposition 7 *Let $L = \operatorname{Zar}(R)$ and $x \in R$. Then $L^{\{\widetilde{x}\}}$ is naturally isomorphic to $\operatorname{Zar}(R^{\{x\}})$ and $L_{\{\widetilde{x}\}}$ is naturally isomorphic to $\operatorname{Zar}(R_{\{x\}})$.*

We get an elementary inductive characterization of Krull dimension of commutative rings. Recall that a ring R has Krull dimension -1 if and only if it is trivial (i.e., $1_R = 0_R$).

Theorem 8 *Let R be a commutative ring and $\ell \geq 0$ an integer. The following are equivalent*

(1) *The Krull dimension of R is $\leq \ell$.*

(2) *For all $x \in R$ the Krull dimension of $R^{\{x\}}$ is $\leq \ell - 1$.*

(3) *For all $x \in R$ the Krull dimension of $R_{\{x\}}$ is $\leq \ell - 1$.*

These equivalences are immediate consequences of Theorem 3 and Proposition 7.

Corollary 9 *(cf. [3, 10]) Let ℓ be a nonnegative integer. The Krull dimension of R is $\leq \ell$ if and only if for all $x_0, \ldots, x_\ell$ in R there exists $a_0, \ldots, a_\ell \in R$ and $m_0, \ldots, m_\ell \in \mathbb{N}$ such that*

$$x_0^{m_0}(\cdots(x_\ell^{m_\ell}(1 + a_\ell x_\ell) + \cdots) + a_0 x_0) = 0 \tag{1}$$

Proof.
Since dimension -1 corresponds to the trivial ring the equivalence for the case $\ell = 0$ is clear.
Assume the equivalence has been established for all integers $< \ell$ and all R. We deduce that the dimension of a localization $S^{-1}R$ is $< \ell$ if and only if for all $x_0, \ldots, x_{\ell-1} \in R$ there exist $a_0, \ldots, a_{\ell-1} \in R$, $s \in S$ and $m_0, \ldots, m_{\ell-1} \in \mathbb{N}$ such that

$$x_0^{m_0}(x_1^{m_1} \cdots (x_{\ell-1}^{m_{\ell-1}}(s + a_{\ell-1}x_{\ell-1}) + \cdots + a_1 x_1) + a_0 x_0) = 0\,. \tag{2}$$

Notice that s replaces 1 in the similar equality (1) with R instead of $S^{-1}R$. It remains only to replace s by an arbitrary element in $S_{\{x_\ell\}}$, i.e., an element $x_\ell^{m_\ell}(1 + a_\ell x_\ell)$. □

The advantage of this definition is, besides its elementary character, to allow simple proofs by induction on the dimension. We can for instance prove directly in this way the following non-Noetherian version of Bass' stable range theorem.

Theorem 10 *If the dimension of R is $< n$ and $1 = D(a, b_1, \ldots, b_n)$ there exists $x_1, \ldots, x_n$ such that $1 = D(b_1 + ax_1, \ldots, b_n + ax_n)$.*

Examples
If $A = \mathbb{Z}$ and $n \neq 0, 1, -1$ then $\mathbb{Z}^{\{n\}} = \mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Z}_{\{n\}} = \mathbb{Q}$. These are two 0-dimensional rings. For $n = 0, 1$ or -1 the two boundaries are trivial. Thus the Krull dimension of $\mathbb{Z}$ is 1.

Let $\mathbb{K}$ be a field contained in an algebraically closed field $\mathbb{L}$, and J be a finitely generated ideal of $\mathbb{K}[X_1, \ldots, X_n]$ and $A = \mathbb{K}[X_1, \ldots, X_n]/J$. If V is the algebraic variety corresponding to J in $\mathbb{L}^n$, if $f \in A$ defines the subvariety W of V and if B is the boundary of W in V, defined as the intersection of W with the Zariski cloture of its complement in V, then the affine variety B corresponds to the ring $A^{\{f\}}$.

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