A NILREGULAR ELEMENT PROPERTY

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Abstract. An element a of a commutative ring R is nilregular if and only if x is nilpotent whenever ax is nilpotent. More generally, an ideal I of R is nilregular if and only if x is nilpotent whenever ax is nilpotent for all $a \in I$. We give a direct proof that if R is Noetherian, then every nilregular ideal contains a nilregular element. In constructive mathematics, this proof can then be seen as an algorithm to produce nilregular elements of nilregular ideals whenever R is coherent, Noetherian, and discrete. As an application, we give a constructive proof of the Eisenbud–Evans–Storch theorem that every algebraic set in n–dimensional affine space is the intersection of n hypersurfaces.

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1. The nilregular element property

Let R be a commutative ring with unit and N its nilradical, i.e. the ideal consisting of its nilpotent elements. We define an element a (respectively, an ideal I) of R to be *nilregular* if and only if $x \in N$ whenever $ax \in N$ (respectively, $ax \in N$ for all $a \in I$). So an ideal I is nilregular precisely when the transporter ideal $(N : I) = \{x \in R : xI \subseteq N\}$ is contained in N. We present a method to find nilregular elements of nilregular ideals when R is Noetherian. For this, we interpret first the property of being nilregular in a topological way.

As usual, let $\mathfrak{D}(a)$ be the set of prime ideals $\mathfrak{p}$ of R such that $a \notin \mathfrak{p}$, and let $\mathfrak{D}(a_1, \dots, a_n)$ stand for the union of $\mathfrak{D}(a_1)$, ..., $\mathfrak{D}(a_n)$. The intersection of $\mathfrak{D}(a)$ and $\mathfrak{D}(b)$ is $\mathfrak{D}(ab)$, and $\mathfrak{D}(a)$ is a subset of $\mathfrak{D}(a_1, \dots, a_n)$ if and only if a belongs to the radical of the ideal $(a_1, \dots, a_n)$ generated by $a_1, \dots, a_n$. In particular, $\mathfrak{D}(a) = \emptyset$ precisely when $a \in N$.

Lemma 1.1. *We have $\mathfrak{D}(a + b, ab) = \mathfrak{D}(a, b)$ for all $a, b \in R$. If, in particular, $\mathfrak{D}(a)$ and $\mathfrak{D}(b)$ are disjoint, then $\mathfrak{D}(a + b) = \mathfrak{D}(a, b)$.*

Proof. Both $a^2 = a(a + b) - ab$ and $b^2 = (a + b)b - ab$ belong to $(a + b, ab)$. □

It is well–known that the $\mathfrak{D}(a)$ with $a \in R$ form a basis of opens for the Zariski topology on the prime spectrum (the set of all prime ideals) of R. It follows that $a \in R$ is nilregular if and only if $\mathfrak{D}(a)$ is dense for the Zariski topology.

Remark 1.2. $\mathfrak{D}(a_1, \dots, a_n)$ *is dense if and only if* $(a_1, \dots, a_n)$ *is a nilregular ideal.*

Theorem 1.3. *Let R be Noetherian, and $a_1, \dots, a_n \in R$. If $\mathfrak{D}(a_1, \dots, a_n)$ is dense, then the ideal $(a_1, \dots, a_n)$ contains a nilregular element.*

Proof. If $\mathfrak{D}(x) \neq \emptyset$, then there exists i such that $\mathfrak{D}(xa_i) \neq \emptyset$, because $\mathfrak{D}(a_1, \dots, a_n)$ is dense. Hence if the ring is nontrivial, then we can inductively build a sequence b_0, b_1, ...

of elements of R in the following way: b_0 is one a_i such that $\mathfrak{D}(b_0) \neq \emptyset$; if $\mathfrak{D}(b_0, \ldots, b_k)$ is not dense, then b_{k+1} is a multiple of one a_j such that $\mathfrak{D}(b_{k+1}) \neq \emptyset$ and $\mathfrak{D}(b_{k+1})$ is disjoint from $\mathfrak{D}(b_0, \ldots, b_k)$. Since R is Noetherian, this procedure has to stop, and we eventually find p such that $\mathfrak{D}(b_0, \ldots, b_p)$ is dense and $\mathfrak{D}(b_i) \cap \mathfrak{D}(b_j) = \emptyset$ whenever $i \neq j$. By Lemma 1.1, we have

$$\mathfrak{D}(b_0, \ldots, b_p) = \mathfrak{D}(b_0 + \cdots + b_p)$$

and $b_0 + \cdots + b_p$ is a nilregular element in $(a_1, \ldots, a_n)$. □

As in [1] we define the *ideal boundary* N_a of $a \in R$ to be the ideal generated by a and the elements x of R such that ax is nilpotent; in other words, $N_a = aR + (N : a)$.

Lemma 1.4. *Every ideal boundary is a nilregular ideal.*

Proof. Fix $a \in R$, and assume that bx is nilpotent for all $b \in N_a$. Then x is nilpotent. Indeed, ax is nilpotent because $a \in N_a$; whence $x \in N_a$ and thus x^2 is nilpotent. □

Corollary 1.5. *If R is Noetherian, then every ideal boundary contains a nilregular element.*

Throughout this section we could only have required that the topological space Spec(R) rather than the ring R be Noetherian.

2. Constructive interpretation

We interpret the previous argument in the framework of constructive mathematics [6, 7]. Let $L(R)$ be the lattice of *radically finitely generated* ideals of R: that is, the radicals of finitely generated ideals [2]. Following Joyal [5], the lattice $L(R)$, with inclusion as ordering, can also be defined as the distributive lattice generated by the symbols $D(a)$ with $a \in R$, and equipped with the relations

$$D(0) = 0\,, \quad D(1) = 1\,, \quad D(ab) = D(a) \wedge D(b)\,, \quad D(a+b) \leq D(a) \vee D(b)$$

for $a, b \in A$. Writing $D(a_1, \ldots, a_m)$ for $D(a_1) \vee \cdots \vee D(a_m)$, it can be shown [2] that

$$D(b_1) \wedge \cdots \wedge D(b_n) \leq D(a_1, \ldots, a_m)$$

if and only if the monoid generated by $b_1, \ldots, b_n$ meets the ideal generated by $a_1, \ldots, a_m$. So $D(a_1, \ldots, a_m)$ can indeed be identified with the radical of the ideal $(a_1, \ldots, a_m)$, and $D\,(a) = 0$ precisely when a is nilpotent.

Lemma 2.1. *If R is coherent, Noetherian, and discrete, then one can decide whether a given element of R is nilpotent.*

Proof. Let $a \in R$. Every annihilator $(0 : a^p)$ is a finitely generated ideal with $(0 : a^p) \subseteq (0 : a^{p+1})$. Since R is Noetherian, there exists n such that $(0 : a^n) = (0 : a^{n+1})$. We even have $(0 : a^n) = \big(0 : a^{n+k}\big)$ for all k. (Indeed, if $a^{n+k+1}b = 0$, then $a^k b$ annihilates a^{n+1} and thus also a^n, so that $a^{n+k}b = 0$.) Hence a is nilpotent if and only if $a^n = 0$. □

Corollary 2.2. *If R is coherent, Noetherian, and discrete, then equality to 0 is decidable in $L\,(R)$.*

We recall that a lattice is a *Heyting algebra* if and only if one can assign to every pair (u, v) of elements another element $u \to v$ such that $u \wedge x \leq v$ if and only if $x \leq u \to v$. In a Heyting algebra, one writes $\neg u$ for $u \to 0$.

If R is coherent and Noetherian, then $L(R)$ is a Heyting algebra [2]. A direct argument shows that $a \in R$ is nilregular if and only if $\neg D(a) = 0$.

Remark 2.3. $\neg D(a_1, \ldots, a_n) = 0$ *if and only if* $(a_1, \ldots, a_n)$ *is a nilregular ideal.*

Lemma 2.4. *If R is coherent, Noetherian, and discrete, for given $b_0, \ldots, b_k \in R$ we can decide whether $\neg D(b_0, \ldots, b_k) = 0$; if indeed $\neg D(b_0, \ldots, b_k) \neq 0$, then we can compute $b_{k+1} \in R$ such that $D(b_{k+1}) \neq 0$ and $D(b_{k+1}) \wedge D(b_0, \ldots, b_k) = 0$.*

Proof. Write $\neg D(b_0, \ldots, b_k) = D(c_1, \ldots, c_m)$, and apply Lemma 2.1 successively to the c_j. If $c_j \notin N$ for some j, then $b_{k+1} = c_j$ is as desired. □

Corollary 2.5. *If R is coherent, Noetherian and discrete, then we can decide whether an element b of R is nilregular, and if this is not the case, then we can compute an element $x \notin N$ such that $bx \in N$.*

In this context, $\neg D(b_0, \ldots, b_k) = 0$ precisely when $D(b_0, \ldots, b_k)$ is dense. Reasoning as in the previous section (Theorem 1.3), we can now conclude.

Theorem 2.6. *Let R be coherent, Noetherian, and discrete, and $a_1, \ldots, a_n \in R$. If $\neg D(a_1, \ldots, a_n) = 0$, then the ideal $(a_1, \ldots, a_n)$ contains a nilregular element.*

This result seems closely connected to the regular element property proved constructively in [7]. The hypothesis is a little weaker (we don't assume the ring to contain an infinite field), but the statement is a priori different unless the ring is reduced (we use 'nilregular' instead of 'regular').

In view of Lemma 1.4, Corollary 1.5 can be rephrased as follows.

Corollary 2.7. *If R is coherent, Noetherian, and discrete, then every ideal boundary contains a nilregular element.*

In terms of $L(R)$, this means that for every $a \in R$ there is $s \in R$ with $\neg D(s) = 0$ and $D(s) \leq D(a) \vee \neg D(a)$; observe that $D(N_a) = D(a) \vee \neg D(a)$.

3. Application

The motivation of this work was to give a constructive proof of the Eisenbud–Evans–Storch theorem that every algebraic set in n–dimensional affine space is the intersection of n hypersurfaces [4, 8]. In [1, 2, 3] a constructive approach to the theory of Krull dimension is given with $\mathsf{Kdim} R \leq -1$ if and only if R is trivial, and $\mathsf{Kdim} R \leq n+1$ if and only if $\mathsf{Kdim}(R/N_a) \leq n$ for all $a \in R$. This inductive definition of being of Krull dimension $\leq n$ is then classically equivalent to the usual definition that there is no strictly increasing chain of prime ideals of length $> n$ [2, 3].

We say that two rings R_1 and R_2 *have the same Krull dimension* if and only if $\mathsf{Kdim} R_1 \leq n$ is equivalent to $\mathsf{Kdim} R_2 \leq n$ for every $n \geq -1$.

Lemma 3.1. *If I is an ideal of R with $I \subseteq N$, then R and R/I have the same Krull dimension.*

Proof. For every $a \in R$, the ideal boundary of the residue class of a in R/I is nothing but N_a/I; moreover, R is trivial precisely when R/I is so. □

In particular, the *reduction* R/N of R has the same Krull dimension as R.

Corollary 3.2. *If* $\mathsf{Kdim} R \leq n+1$ *and $s \in R$ is nilregular, then* $\mathsf{Kdim}(R/sR) \leq n$

Proof. In this case we have $N_s = sR + N$, and $R/\left(sR+N\right)$ is the reduction of R/sR. □

Lemma 3.3. *If R is reduced, then R is von Neumann regular if and only if* $\mathsf{Kdim} R \leq 0$.

Proof. By definition $\mathsf{Kdim} R \leq 0$ if and only if for every a there exists x such that $a(1-xa) = 0$, which means that R is von Neumann regular. □

Corollary 3.4. *If R is reduced and* $\mathsf{Kdim} R \leq 0$*, then every finitely generated ideal of $R[X]$ is principal. If we assume only* $\mathsf{Kdim} R \leq 0$*, then every radically finitely generated ideal of $R[X]$ is radically generated by one element.*

Proof. It is a standard argument that if R is von Neumann regular, then every finitely generated ideal of $R[X]$ is principal. □

We call a ring R *strongly discrete* if and only if we can decide whether $a \in I$ for each finitely generated ideal I of R and every $a \in R$. Clearly, R is strongly discrete precisely when R/I is discrete for every finitely generated ideal I of R.

Theorem 3.5. *Let R be coherent, Noetherian, and strongly discrete. If* $\mathsf{Kdim} R \leq d$*, then for every $g_1, \ldots, g_m \in R[X]$ there exists $f_0, \ldots, f_d \in R[X]$ such that $D(g_1, \ldots, g_m) = D(f_0, \ldots, f_d)$.*

Proof. We prove this by induction on d. The statement is clear from Corollary 3.4 if $d = 0$.

Let S be the multiplicative monoid of nilregular elements. Corollary 2.7 shows that the ring of fractions R_S is of Krull dimension ≤ 0. Hence, using Corollary 3.4 again, we can find $f \in R[X]$ such that $D(f) = D(g_1, \ldots, g_m)$ in $R_S[X]$. In $R[X]$ this means that there exists $s \in S$ such that

$$D(f) \wedge D(s) \leq D(g_1, \ldots, g_m) \quad \text{and} \quad D(g_i) \wedge D(s) \leq D(f)\,.$$

We now set $f_0 = sf$ and thus arrive at

$$D(s) \wedge D(g_1, \ldots, g_m) \leq D(f_0) \leq D(g_1, \ldots, g_m)$$

in $R[X]$.

Since $s \in S$, we have $\mathsf{Kdim}(R/sR) \leq d-1$ by Corollary 3.2. By induction, we can find $h_1, \ldots, h_d$ such that

$$D(h_1, \ldots, h_d) = D(g_1, \ldots, g_m)$$

in $(R/sR)[X]$. (Induction is possible, because if R is coherent, Noetherian, and strongly discrete, then so is R/I for every finitely generated ideal I of R [6, III.2].) This means

$$D(s, h_1, \ldots, h_d) = D(s, g_1, \ldots, g_m)$$

in $R[X]$; whence $h_j^{n_j} = a_j s + \sum_i c_{ij} g_i$ for $j = 1, ..., d$ and suitable integers $n_j \geq 1$.

For each $j \geq 1$, we now set $f_j = h_j^{n_j} - a_j s$ and get $D(f_j, s) = D(h_j, s)$ with $D(f_j) \leq D(g_1, ..., g_m)$. This gives

$$D(s, f_1, ..., f_d) = D(s, g_1, ..., g_m)$$

and thus $D(f_0, f_1, ..., f_d) \leq D(g_1, ..., g_m)$. For each $i \leq m$, moreover, $D(g_i) \leq D(s, f_1, ..., f_d)$ implies $D(g_i) \leq D(sg_i, f_1, ..., f_d)$; since, in addition, $D(sg_i) \leq D(f_0)$, we get $D(g_i) \leq D(f_0, f_1, ..., f_d)$.We finally arrive at $D(f_0, f_1, ..., f_d) = D(g_1, ..., g_m)$ as desired. □

To prove Theorem 3.5 in this way, by induction on the Krull dimension of R, we apply Corollary 3.4 not only to R, but also to certain quotient rings of R (for instance, to R/sR for some nilregular element s). Hence we need to know that *all* these rings are discrete, which is guaranteed by the assumption that R be *strongly* discrete. Note that if R is coherent, Noetherian, and strongly discrete, then $L(R)$ is discrete [2].

In [2] it is shown, in an elementary and constructive way, that the Krull dimension of a polynomial ring in n variables over a discrete field is $\leq n$. By the constructive version of Hilbert's basis theorem [6, VIII.1.5], any such polynomial ring is coherent, Noetherian, and strongly discrete.

Corollary 3.6. *If K is a discrete field and $d \geq 1$, then for all $g_1, \ldots, g_m \in K[X_1, \ldots, X_d]$ there exist $f_1, \ldots, f_d \in K[X_1, \ldots, X_d]$ such that $D(g_1, \ldots, g_m) = D(f_1, \ldots, f_d)$.*

Kronecker proved this result with $d + 1$ polynomials instead of d polynomials [1]. Our argument, being constructive, can be read as an algorithm that produces $f_1, \ldots, f_d$ for given $g_1, \ldots, g_m$.

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