The projective spectrum as a distributive lattice

Thierry Coquand, Henri Lombardi, and Peter Schuster

Résumé

Nous construisons un treillis distributif dont les filtres premiers correspondent aux idéaux premiers homogènes d'un anneau commutatif gradué. Ceci donne un exemple charactéristique d'un schéma non affine en topologie sans points, et d'une construction générale de recollements de treillis distributifs. Nous prouvons aussi une forme projective du "théorème des zeros" d'Hilbert.

Abstract

We construct a distributive lattice whose prime filters correspond to the homogeneous prime ideals of a graded commutative ring. This is the prime example of a non–affine scheme in a point–free context, and the model case of a fairly general glueing method for distributive lattices. We furthermore prove the projective form of the formal Hilbert Nullstellensatz.

A *formal point* of a distributive lattice T is a *prime filter* of T: that is, a subset F of T with

$$\begin{array}{l} 1 \in F \\ x \wedge y \in F \iff x \in F \wedge y \in F \\ 0 \notin F \\ x \vee y \in F \Longrightarrow x \in F \vee y \in F \end{array} \tag{1}$$

for all $x, y \in T$. We write $\mathrm{Pt}\,(T)$ for the space of formal points of T; the family $\{F \in \mathrm{Pt}\,(T) : x \in F\}$ with $x \in T$ is a basis of open subsets for the topology on $\mathrm{Pt}\,(T)$.

Lemma 1 *Let T be a distributive lattice, and $z \in T$. Consider the quotient T' of T modulo $z = 1$. For the projection mapping $\pi_z : T \to T'$ we have*

$$\pi_z\,(x) \leqslant \pi_z\,(y) \iff x \wedge z \leqslant y \wedge z$$

for all $x, y \in T$. In particular, T' can be identified with $\downarrow z$ and π_z with the mapping $x \mapsto x \wedge z$, where $\downarrow z$ has the lattice structure induced by that on T with the only exception that z stands for 1 in $\downarrow z$.

In particular, $\mathrm{Pt}\,(T')$ is an open subspace of $\mathrm{Pt}\,(T)$, with the inclusion mapping $\mathrm{Pt}\,(T') \to \mathrm{Pt}\,(T)$ as the one induced by π_z. For short, $\mathrm{Pt}\,(\pi_z) : \mathrm{Pt}\,(T') \to \mathrm{Pt}\,(T)$ is an *open inclusion.*

Joyal [4] presented the affine spectrum

$$\mathrm{Spec}(A) = \{\mathfrak{p} \subset A : \mathfrak{p} \text{ prime ideal of } A\}$$

of a commutative ring A in a point–free way as the distributive lattice $L(A)$ that is generated by the expressions of the form $D_L(a)$ with $a \in A$, and equipped with the relations

$$\begin{array}{l} D_L(1) = 1 \\ D_L(ab) = D_L(a) \wedge D_L(b) \\ D_L(0) = 0 \\ D_L(a+b) \leqslant D_L(a) \vee D_L(b) \end{array} \tag{2}$$

for all $a, b \in A$. The intuition standing behind this is that in terms of points the family

$$D(a) = \{\mathfrak{p} \in \operatorname{Spec}(A) : a \in A \setminus \mathfrak{p}\} \qquad (a \in A)$$

is a basis of open subsets for the Zariski topology on $\operatorname{Spec}(A)$, whose characteristic properties are expressed by (2) with $\subseteq$, $\emptyset$, and $\operatorname{Spec}(A)$ in place of $\leqslant$, 0, and 1, respectively. In fact, the formal points of $L(A)$ are nothing but the prime filters of A: that is, the subsets F of A which satisfy (1) with addition and multiplication in place of $\vee$ and $\wedge$.

When one admits reasoning by classical logic, the prime filters of A are the complements of the prime ideals of A; then $\operatorname{Pt}(L(A))$ and $\operatorname{Spec}(A)$ are even homeomorphic. From the constructive point of view, however, prime filters are to be preferred: 'because it is at these objects that we wish to localize, and since $\neg\neg \neq \mathrm{id}$, we must deal with them directly' [5, p. 194].

For a commutative ring A and $a \in A$, we denote by $A\left[\frac{1}{a}\right]$ the ring of fractions whose denominators are the powers of a, which ring is isomorphic to $A[X]\,/\,(aX-1)$.

Lemma 2 *Let A be a commutative ring and $a \in A$. Then the quotient of $L(A)$ modulo $D_L(a) = 1$ can be identified with $L\left(A[\frac{1}{a}]\right)$; moreover, the projection mapping $\pi_{D_L(a)} : L(A) \to L\left(A[\frac{1}{a}]\right)$ is induced by the canonical mapping $A \to A[\frac{1}{a}]$.*

In particular, $A \to A[\frac{1}{a}]$ induces an open inclusion $\operatorname{Pt}(L\left(A[\frac{1}{a}]\right)) \to \operatorname{Pt}(L(A))$.

We now transfer Joyal's approach to the projective spectrum of a graded commutative ring

$$A = \bigoplus_{d \geqslant 0} A_d\,,$$

for which we make the standard assumption (see, for instance, [2]) that A is generated as an A_0–algebra by finitely many $x_0, \ldots, x_n \in A_1$ with $n \geqslant 1$: that is,

$$A = A_0[x_0, \ldots, x_n]\,.$$

A homogeneous prime ideal of A is a prime ideal which

- is generated by homogeneous elements (or, equivalently, a prime ideal that contains an element of the ring precisely when it contains all of its homogeneous components), and
- does not contain the whole of
$$A^+ = \bigoplus_{d>0} A_d$$
(that is, it does not contain all the x_i).

With the family

$$D(a) = \{\mathfrak{p} \in \mathrm{Proj}(A) : a \in A \setminus \mathfrak{p}\} \qquad (a \in A^d,\ d > 0)$$

as a basis of open subsets, the projective spectrum

$$\mathrm{Proj}(A) = \{\mathfrak{p} \subset A : \mathfrak{p} \text{ homogeneous prime ideal of } A\}$$

of A is homeomorphic to the result of glueing together the affine spectra Spec $\left(A[\frac{1}{x_i}]_0\right)$.

The prime example of a graded ring is the ring of polynomials

$$A = k[x_0, \ldots, x_n]\,,$$

graded by degree, in $n+1$ indeterminates $x_0, \ldots, x_n$ with coefficients in a discrete commutative ring k. The projective spectrum

$$\mathbb{P}^n_k = \mathrm{Proj}\,(k[x_0, \ldots, x_n])$$

with an appropriate structure sheaf is the projective scheme of dimension n over k.

Consider the distributive lattice $P\,(A)$ which is generated by the expressions $D_P\,(a)$ with a being a homogeneous element of A^+ (that is, $a \in A_d$ for some $d > 0$), and subject to the relations

$$\begin{array}{l} D_P\,(x_0) \vee \ldots \vee D_P\,(x_n) = 1 \\ D_P(ab) = D_P(a) \wedge D_P(b) \\ D_P(0) = 0 \\ D_P(a+b) \leqslant D_P(a) \vee D_P(b) \end{array} \tag{3}$$

for all homogeneous $a, b \in A^+$. In the last relation, a and b have to have the same degree, to ensure that also $a+b$ is homogeneous.

Since each element of $A[\frac{1}{x_i}]_0$ can be written in the form $\frac{a}{x_j^d}$ with $a \in A$ a homogeneous element of degree $d > 0$, we have the following.

Proposition 3 *Let A be a graded ring as above. For every $i \in \{0, \ldots, n\}$ the quotient of $P\,(A)$ modulo $D_P\,(x_i) = 1$ is isomorphic to $L\left(A[\frac{1}{x_i}]_0\right)$, where the isomorphism is well–defined on the generators by assigning $D_P\,(a)$ and $D_L\left(\frac{a}{x_j^d}\right)$ to each other for every $a \in A_d$ with $d > 0$.*

Moreover, for each pair i, j the diagram

$$\begin{array}{ccc} P\,(A) & \rightarrow & L\left(A\left[\frac{1}{x_i}\right]_0\right) \\ \downarrow & & \downarrow \\ L\left(A\left[\frac{1}{x_j}\right]_0\right) & \rightarrow & L\left(A\left[\frac{1}{x_i}, \frac{1}{xj}\right]_0\right) \end{array}$$

is commutative. Since also

$$A\left[\frac{1}{x_i}\right]_0\left[\left(\frac{x_j}{x_i}\right)^{-1}\right] \cong A\left[\frac{1}{x_i}, \frac{1}{xj}\right]_0 \cong A\left[\frac{1}{x_j}\right]_0\left[\left(\frac{x_i}{x_j}\right)^{-1}\right],$$

all the arrows that occur in this diagram induce open inclusions on the level of points.

From the perspective of classical point–set topology, we now already know that $\mathrm{Pt}\,(P\,(A))$ is homeomorphic to the topological space obtained by glueing together the $\mathrm{Pt}\left(L\left(A[\frac{1}{x_i}]_0\right)\right)$, which correspond to open subspaces. In particular, $\mathrm{Pt}\,(P\,(A))$ is homeomorphic to $\mathrm{Proj}\,(A)$. In constructive point–free topology, however, enough prime filters to constitute these spaces are not always at our disposal, so in every topos which lacks the appropriate form of the axiom of choice.

Lemma 4 (Glueing together finitely many distributive lattices) *Let $L_0, \ldots, L_n$ be distributive lattices with $n \geqslant 1$, distinguished elements $u_{ij} \in L_i$, and lattice isomorphisms $\varphi_{ij} : \downarrow u_{ij} \to \downarrow u_{ji}$ for all i, j. Assume that*

$$u_{ii} = 1 \quad \textit{and} \quad \varphi_{ii} = \mathrm{id}_{L_i}$$

for every i, and that

$$\varphi_{ij}(u_{ij} \wedge \varphi_{ki}\,(u_{ki} \wedge v)) = u_{ji} \wedge \varphi_{kj}\,(u_{kj} \wedge v) \tag{4}$$

for every triple i, j, k and all $v \in L_k$.

1. *With componentwise operations,*

   $$L = \{(v_0, \ldots, v_n) \in L_0 \times \ldots \times L_n : \varphi_{ij}(u_{ij} \wedge v_i) = u_{ji} \wedge v_j \textit{ for all } i, j\}$$

   is a distributive lattice. For each k, if we set

   $$u_k = (u_{0k}, \ldots, u_{nk})\,,$$

   then $u_k \in L$, and the projection mapping $\lambda_k : L \to L_k$ is a lattice homomorphism which induces an isomorphism $\downarrow u_k \cong L_k$. Moreover, we have

   $$u_0 \vee \ldots \vee u_n = 1\,.$$

2. *Let M be a distributive lattice. If for each k there is a lattice homomorphism $\mu_k : M \to L_k$ such that*

   $$\varphi_{ij}\,(u_{ij} \wedge \mu_i\,(w)) = u_{ji} \wedge \mu_j\,(w)$$

   for every pair i, j and all $w \in M$, then

   $$\mu : M \to L\,,\;\; w \mapsto (\mu_0\,(w)\,, \ldots, \mu_n\,(w))$$

   is the unique lattice homomorphism with $\lambda_k \circ \mu = \mu_k$ for every k.

Proof. The only perhaps nontrivial issue is to show that the mapping $\downarrow u_k \to L_k$ induced by λ_k is bijective—or, equivalently, that for each $w \in L_k$ there is precisely one element v of $\downarrow u_k$ with k–th component w. To see this, set $v_i = \varphi_{ki}(u_{ki} \wedge w)$ for every i, and define $v = (v_0, \ldots, v_n)$. Then $v \in L$ because

$$\varphi_{ij}(u_{ij} \wedge v_i) = \varphi_{ij}\,(u_{ij} \wedge \varphi_{ki}(u_{ki} \wedge w)) = u_{ji} \wedge \varphi_{kj}\,(u_{kj} \wedge w) = u_{ji} \wedge v_j$$

by virtue of (4), and $v_i \in \downarrow u_{ik}$ for every i by definition. Hence $v \in \downarrow u_k$; clearly, $v_k = w$. If also $v' = (v'_0, \ldots, v'_n) \in \downarrow u_k$ with $v'_k = w$, then

$$v'_i = u_{ik} \wedge v'_i = \varphi_{ki}(u_{ki} \wedge v'_k) = \varphi_{ki}(u_{ki} \wedge w) = v_i$$

for every i. Q.E.D.

One gets a given distributive lattice back when one glues together finitely many quotients modulo $u_i = 1$.

Lemma 5 *Let K be a distributive lattice with distinguished elements $u_0, \ldots, u_n \in K$ for $n \geqslant 1$. Set $u_{ij} = u_j \wedge u_i$ for every pair i, j. With $L_k = \downarrow u_k$ for every k and $\varphi_{ij} : \downarrow u_{ij} \to \downarrow u_{ji}$ as the identity mapping for every pair i, j, the hypotheses of Lemma 4 are satisfied. Moreover, if $u_0 \vee \ldots \vee u_n = 1$, then $K \cong L$ where L is as in Lemma 4.*

Proof. The lattice homomorphism $K \to L$ with $w \mapsto (w \wedge u_0, \ldots, w \wedge u_n)$ is well–defined (because $u_{ij} \wedge w \wedge u_i = u_{ji} \wedge w \wedge u_j$ for all i, j), and invertible with inverse mapping $L \to K$ defined by $(v_0, \ldots, v_n) \mapsto v_0 \vee \ldots \vee v_n$. In fact, if $w \in K$, then

$$(w \wedge u_0) \vee \ldots \vee (w \wedge u_n) = w \wedge (u_0 \vee \ldots \vee u_n) = w \wedge 1 = w\,.$$

If, on the other hand, $(v_0, \ldots, v_n) \in L$, then $v_i = v_i \wedge u_i$ and $v_i \wedge u_{ij} = v_j \wedge u_{ji}$; whence $v_i \wedge u_j = v_j \wedge u_i$ for all i, j and thus

$$(v_0 \vee \ldots \vee v_n) \wedge u_j = v_j \wedge (u_0 \vee \ldots \vee u_n) = v_j \wedge 1 = v_j$$

for every j. Q.E.D.

By Proposition 3 and Lemma 5, we eventually know that we are doing the right thing.

Proposition 6 *Let A be a graded ring as above. Then $P(A)$ is isomorphic to the distributive lattice which is obtained by glueing together the $n+1$ distributive lattices $L\left(A[\frac{1}{x_i}]_0\right)$.*

The formal affine Hilbert Nullstellensatz [3, V.3.2] says that if $a_1, \ldots, a_k$ and $b_1, \ldots, b_\ell$ are elements of a commutative ring A, then

$$D_L(a_1) \wedge \ldots \wedge D_L(a_k) \leqslant D_L(b_1) \vee \ldots \vee D_L(b_\ell)$$

holds in $L(A)$ if and only if the multiplicative monoid $\langle a_1, \ldots, a_k \rangle$ generated by $a_1, \ldots, a_k$ meets the ideal $(b_1, \ldots, b_\ell)$ generated by $b_1, \ldots, b_\ell$.

Theorem 7 (Formal projective Hilbert Nullstellensatz) *If A is a graded ring as above, and $a_1, \ldots, a_k$, $b_1, \ldots, b_\ell$ are homogeneous elements of A^+, then*

$$D_P(a_1) \wedge \ldots \wedge D_P(a_k) \leqslant D_P(b_1) \vee \ldots \vee D_P(b_\ell)$$

holds in $P(A)$ if and only if $\langle x_i, a_1, \ldots, a_k \rangle$ meets $(b_1, \ldots, b_\ell)$ for every $i \in \{0, \ldots, n\}$.

Proof. Since $D_P(a_1) \wedge \ldots \wedge D_P(a_k) = D_P(a_1 \cdot \ldots \cdot a_k)$, we may assume that $k = 1$; whence we have to show

$$D_P(a) \leqslant D_P(b_1) \vee \ldots \vee D_P(b_\ell) \iff \forall i \left(ax_i \in \sqrt{(b_1, \ldots, b_\ell)}\right)$$

for all homogeneous $a, b_1, \ldots, b_\ell \in A^+$. For each i, if $ax_i \in \sqrt{(b_1, \ldots, b_\ell)}$, then

$$D_P(a) \wedge D_P(x_i) = D_P(ax_i) \leqslant D_P(b_1) \vee \ldots \vee D_P(b_\ell)$$

according to (3)—in fact, one only needs the three relations which are the same for $L(A)$ and $P(A)$. Hence

$$D_P(a) = D_P(a) \wedge (D_P(x_0) \vee \ldots \vee D_P(x_n)) \leqslant D_P(b_1) \vee \ldots \vee D_P(b_\ell)$$

because of the extra relation $D_P(x_0) \vee \ldots \vee D_P(x_n) = 1$ valid only in $P(A)$.

Conversely, assume that $D_P(a) \leqslant D_P(b_1) \vee \ldots \vee D_P(b_\ell)$ holds in $P(A)$, and fix i. Modulo $D_P(x_i) = 1$ and in view of Proposition 3, this amounts to

$$D_L\left(\frac{a}{x_i^d}\right) \leqslant D_L\left(\frac{b_1}{x_i^{e_1}}\right) \vee \ldots \vee D_L\left(\frac{b_\ell}{x_i^{e_\ell}}\right) \quad \text{in} \quad L\left(A\left[\frac{1}{x_i}\right]_0\right)$$

whenever $a \in A_d$ and $b_\nu \in A_{e_\nu}$ for every ν. By the formal affine Hilbert Nullstellensatz, the latter means

$$\frac{a}{x_i^d} \in \sqrt{\left(\frac{b_1}{x_i^{e_1}}, \ldots, \frac{b_\ell}{x_i^{e_\ell}}\right)} \quad \text{in} \quad A\left[\frac{1}{x_i}\right]_0,$$

which is equivalent to $ax_i \in \sqrt{(b_1, \ldots, b_\ell)}$ as required. Q.E.D.

Note that this proof works by applying the formal affine Nullstellensatz to every affine component.

For each commutative ring A, Joyal's $L(A)$ is isomorphic to the distributive lattice of the radicals of finitely generated ideals—with the ordering given by inclusion, join $\sqrt{\mathfrak{a}} \vee \sqrt{\mathfrak{b}} = \sqrt{\mathfrak{a}+\mathfrak{b}}$, and meet $\sqrt{\mathfrak{a}} \wedge \sqrt{\mathfrak{b}} = \sqrt{\mathfrak{a}\cdot\mathfrak{b}}$. This was deduced in [1] from the formal affine Hilbert Nullstellensatz. Reasoning in an analogous way, one can draw the following consequence from Theorem 7.

Corollary 8 *If A is a graded ring as above, then $P(A)$ is isomorphic to the quotient modulo $\sqrt{(x_0, \ldots, x_n)} = 1$ of the distributive lattice formed by the radicals of finitely generated ideals whose generators are homogeneous of positive degree.*

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Thierry Coquand, Chalmers Institute of Technology and University of Göteborg, Sweden; `coquand@cs.chalmers.se`

Henri Lombardi, Equipe de Mathématiques, Université de Franche–Comté, 25030 Besançon cedex, France; `henri.lombardi@univ-fcomte.fr`

Peter Schuster, Mathematisches Institut, Universität München, Theresienstraße 39, 80333 München, Germany; `Peter.Schuster@mathematik.uni-muenchen.de`