Spectral Schemes as Ringed Lattices

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"What would have happened if topologies *without* points had been discovered before topologies *with* points, or if Grothendieck had known the theory of distributive lattices?"

Gian-Carlo Rota, *Indiscrete Thoughts.* Birkhäuser (1997), p. 220

Abstract

We give a point-free definition of a Grothendieck scheme whose underlying topological space is spectral. Affine schemes aside, the prime examples are the projective spectrum of a graded ring and the space of valuations corresponding to an abstract nonsingular curve. With the appropriate notion of a morphism between spectral schemes, elementary proofs of the universal properties become possible.

1 Introduction

A partial realisation of Hilbert's programme has recently proved successful in commutative algebra [2, 3, 5, 4, 6, 7, 8, 9, 10, 11, 12, 13, 14, 16, 17, 18, 27, 28, 29, 30, 31, 33, 38, 46]. One of the key tools is Joyal's point-free version of the Zariski spectrum as a distributive lattice [26], studied further by Español [21] and later taken up [22, 42, 43, 45] in the context of formal topology [40, 41]. To extend this in the direction of an analogous treatment of algebraic geometry, Grothendieck's language of schemes [23] needs to be reformulated with, in Hilbert's sense, finite methods. It turns out that distributive lattices even suffice for representing all the schemes whose underlying topological spaces are spectral, which

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includes pivotal cases such as the projective spectrum of a graded ring [15] and the space of valuations corresponding to an abstract nonsingular curve [4].

The paper is written in the tradition of constructive algebra [35, 32]. Each of our statements can be understood as the specification of a program, and its proof can be seen as a program realising this specification together with a proof of correctness. One motivation of this work is to understand the computational content of abstract methods in algebraic geometry; we also think that our work can help to formalise algebraic geometry (see [1] for an attempt of formalising Grothendieck's notion of a scheme).

Every ring is assumed to be commutative with unit, and every lattice to be distributive and bounded. Note that distributivity will be essential for sheaves on lattices.

2 Basic Definitions

The following definitions are more-or-less standard in the category-theoretic literature, see e.g. [34]. They moreover correspond to a special case, the one of finitary formal topologies, of a set of concepts [43] given in the context of formal topology [40, 41]. In the more elementary context of sheaves on distributive lattices, however, we can proceed in a considerably simpler fashion.

2.1 The Zariski Lattice

Joyal [26] (see also Español [21]) presented the affine spectrum[1]

$$\mathfrak{Spec}(A) = \{\mathfrak{p} \subseteq A : \mathfrak{p} \text{ prime ideal of } A\}$$

of a ring A in a point-free way as the lattice L_A generated by the symbols $D(a)$ with $a \in A$ which are subject to the relations

$$\begin{aligned} &D(1) = 1 \\ &D(ab) = D(a) \wedge D(b) \\ &D(0) = 0 \\ &D(a+b) \leqslant D(a) \vee D(b) \end{aligned} \tag{1}$$

for all $a, b \in A$. The intuition standing behind the choice of L_A is that, in terms of points, the family

$$\mathfrak{D}(a) = \{\mathfrak{p} \in \mathfrak{Spec}(A) : a \in A \setminus \mathfrak{p}\} \qquad (a \in A)$$

is a basis of open subsets for the Zariski topology on $\mathfrak{Spec}(A)$, whose characteristic properties are expressed by (1) with $\subseteq$, $\emptyset$, and $\mathfrak{Spec}(A)$ in place of $\leqslant$, 0, and 1, respectively. Some more details on the representation of $\mathfrak{Spec}(A)$ by L_A can be found in, e.g., [15].

A *support* [26] on a commutative ring A with values in a lattice L is a mapping $d : A \to L$ that satisfies (1) with d in place of D. If $d : A \to L$ is a support, then so is $f \circ d \circ \psi$ whenever

[1] To distinguish spaces consisting of points from their point-free counterparts, we use Gothic type for the former and Roman type for the latter.

$\psi : A' \to A$ and $f : L \to L'$ are homomorphisms of rings and lattices, respectively. The support $D : A \to L_A$ which assigns $D(a) \in L_A$ to $a \in A$ has the universal property that if L is a lattice and $d : A \to L$ a support, then there is exactly one lattice homomorphism $f : L_A \to L$ with $f \circ D = d$. In particular, for every ring homomorphism $\psi : A \to B$ there is exactly one lattice homomorphism $f_\psi : L_A \to L_B$ with $f_\psi \circ D = D \circ \psi$.

The elements of L_A are of the form

$$D(a_1, \ldots, a_n) = D(a_1) \vee \ldots \vee D(a_n)$$

with $a_1, \ldots, a_n \in A$. With $D(a_1, \ldots, a_n)$ corresponding to $\sqrt{(a_1, \ldots, a_n)}$, the Zariski lattice $L(A)$ is isomorphic to the lattice of the radicals of finitely generated ideals whose ordering is given by inclusion and whose join and meet are as follows:

$$\sqrt{I} \vee \sqrt{J} = \sqrt{I + J}\,, \quad \sqrt{I} \wedge \sqrt{J} = \sqrt{I \cdot J}\,.$$

In particular, there is the so-called formal Hilbert Nullstellensatz [8, 25]:

$$D(a_1, \ldots, a_n) \leqslant D(b_1, \ldots, b_m) \iff \sqrt{(a_1, \ldots, a_n)} \subseteq \sqrt{(b_1, \ldots, b_m)}\,. \tag{2}$$

As the counterpart of the structure sheaf of $\mathfrak{Spec}(A)$ we next conceive a sheaf of rings on the lattice L_A. This requires us to recall sheaves on lattices first.

2.2 Sheaves on Lattices

As usual, a poset L is understood as the small category in which there only are the morphisms $x \leqslant y$ with $x, y \in L$; whence in the dual poset L^{op} there is a morphism from x to y precisely when $x \geqslant y$.

Definition 1 (presheaf) *A* presheaf *on a poset L with values in a category C is a functor $\mathcal{F} : L^{\mathrm{op}} \to C$.*

In other words, a presheaf on a poset L consists of objects $\mathcal{F}(x)$ of C with $x \in L$ and restriction morphisms $\mathcal{F}(x \leqslant y) : \mathcal{F}(y) \to \mathcal{F}(x)$ in C with $x \leqslant y$ such that $\mathcal{F}(x \leqslant x) = \mathrm{id}_{\mathcal{F}(x)}$ and $\mathcal{F}(x \leqslant z) = \mathcal{F}(x \leqslant y) \circ \mathcal{F}(y \leqslant z)$ whenever $x \leqslant y \leqslant z$.

Note that every subset L' of a poset L defines a full subcategory. In this context, if $\mathcal{F}$ is a presheaf on L, then $\mathcal{F}|_{L'}$ denotes the restriction of $\mathcal{F}$ to L'.

Convention In the following we will only consider presheaves with values in a fixed category C with finite inverse limits—or, equivalently, with finite products (that is, with a terminal object and binary products) and equalisers. When it comes to talk about sheaves, the lattices under consideration need to be distributive.

Definition 2 (sheaf) *A presheaf $\mathcal{F}$ on a distributive lattice L is a* sheaf *if*

$$\mathcal{F}(x_1 \vee \ldots \vee x_n) = \varprojlim \{\mathcal{F}(x_i) \to \mathcal{F}(x_i \wedge x_j) : 1 \leqslant i, j \leqslant n\,,\ i \neq j\} \tag{3}$$

for all $x_1, \ldots, x_n \in L$ with $n \geqslant 0$.

In other words, a presheaf $\mathcal{F}$ on a lattice L is a sheaf if and only if $\mathcal{F}(0) = 0$ and

$$\begin{array}{ccc} \mathcal{F}(x \vee y) & \to & \mathcal{F}(y) \\ \downarrow & & \downarrow \\ \mathcal{F}(x) & \to & \mathcal{F}(x \wedge y) \end{array}$$

is a pullback diagramme for all $x, y \in L$. Moreover, in L^{op} we have

$$x_1 \vee \ldots \vee x_n = \varprojlim \{x_i \geqslant x_i \wedge x_j : 1 \leqslant i, j \leqslant n\,,\; i \neq j\}\;;$$

whence (3) says that $\mathcal{F}$ preserves finite inverse limits of this particular kind. Also, if $\mathcal{F}$ is a sheaf on a lattice L and L' a sublattice of L, then $\mathcal{F}|_{L'}$ is a sheaf on L'.

Definition 3 (morphism of (pre)sheaves) *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be (pre)sheaves on L with values in C. A* morphism of (pre)sheaves $\mathcal{F}_1 \to \mathcal{F}_2$ *is a natural transformation $\varphi : \mathcal{F}_1 \to \mathcal{F}_2$.*

The presheaves on L with values in C form a category, of which the sheaves form a full subcategory.

We next adapt to the setting of lattices the familiar method to extend a sheaf on a basis to one on the whole topology [34, pp. 69, 589]. For this purpose we understand by a *basis* of a lattice L a subset L' which is closed under finite meets (that is, $1 \in L'$, and $x \wedge y \in L'$ whenever $x \in L'$ and $y \in L'$) and for which every element of L is a finite join of elements of L'. For example, if A is a ring, then the $D(a)$ with $a \in A$ form a basis of L_A.

Let L' be a basis of a distributive lattice L. A *sheaf* on L' is a presheaf $\mathcal{F}$ on L' such that (3) holds for all $x_1, \ldots, x_n \in L'$ with $x_1 \vee \ldots \vee x_n \in L'$. The category of sheaves on L' is to be a full subcategory of the category of presheaves on L'. If $\mathcal{F}$ is a sheaf on L, then $\mathcal{F}|_{L'}$ is a sheaf on L', and likewise for morphisms.

The restriction functor from the category of sheaves on L to the category of sheaves on L' is an equivalence. This follows from the "comparison lemma" [34, Appendix, Corollary 3]; we now give an elementary proof.

Lemma 4 *Let L' be a basis of a lattice L. For each sheaf $\mathcal{F}'$ on L' there is a sheaf $\mathcal{F}$ on L with $\mathcal{F}|_{L'} \cong \mathcal{F}'$ which by this condition is determined up to unique isomorphism. Moreover, if $\mathcal{F}$ and $\mathcal{G}$ are sheaves on L, then for every morphism of sheaves $\varphi' : \mathcal{F}|_{L'} \to \mathcal{G}|_{L'}$ there is a uniquely determined morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ such that $\varphi'(x) = \varphi(x)$ for all $x \in L'$.*

Proof. For every $x \in L$ choose $x_1, \ldots, x_n \in L'$ with $x = x_1 \vee \ldots \vee x_n$ in L and set

$$\mathcal{F}(x) = \varprojlim \{\mathcal{F}'(x_i) \to \mathcal{F}'(x_i \wedge x_j) : 1 \leqslant i, j \leqslant n\,,\; i \neq j\}\,. \tag{4}$$

It is routine to verify that this definition of $\mathcal{F}(x)$ is independent up to unique isomorphism of the choice of the $x_1, \ldots, x_n$. In particular, $\mathcal{F}(x) \cong \mathcal{F}'(x)$ whenever $x \in L'$. Assume now that

$$x = x_1 \vee \ldots \vee x_n \leqslant y_1 \vee \ldots \vee y_m = y$$

in L where $x_1, \ldots, x_n \in L'$ and $y_1, \ldots, y_m \in L'$. For every $k \leqslant n$ we have

$$x_k = (x_k \wedge y_1) \vee \ldots \vee (x_k \wedge y_m)$$

and thus

$$\mathcal{F}(x_k) = \varprojlim \{\mathcal{F}'(x_k \wedge y_i) \to \mathcal{F}'(x_k \wedge y_i \wedge y_j) : 1 \leqslant i, j \leqslant m\,,\ i \neq j\}\ ;$$

whence the compositions $\mathcal{F}(y) \to \mathcal{F}'(y_i) \to \mathcal{F}'(x_k \wedge y_i)$ induce a uniquely determined arrow $\mathcal{F}(y) \to \mathcal{F}(x_k)$ that commutes with the restriction morphisms $\mathcal{F}(x_k) \to \mathcal{F}'(x_k \wedge y_i)$. These arrows $\mathcal{F}(y) \to \mathcal{F}(x_k)$ induce a uniquely determined morphism from $\mathcal{F}(y) \to \mathcal{F}(x)$ which commutes with the restriction morphisms $\mathcal{F}(x) \to \mathcal{F}(x_k)$. Standard computations show that $\mathcal{F}$ is a sheaf on L. Along similar lines one sees the only possible way to extend a morphism of sheaves $\varphi' : \mathcal{F}|_{L'} \to \mathcal{G}|_{L'}$ to a morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$. Q.E.D.

Note that for every $x \in L$ there is a canonical isomorphism

$$\mathcal{F}(x) \cong \varprojlim \{\mathcal{F}'(v) \to \mathcal{F}'(u) : u \leqslant v \leqslant x;\ u, v \in L'\}\ . \tag{5}$$

Using (5) in place of (4) already to define $\mathcal{F}(x)$ would require to speak of potentially infinite diagrams, but would relieve us from the task to prove independence of any choice of data. Moreover, it would facilitate the definition of the restriction morphism: if $x \leqslant y$ in L, then the diagramme in (5) for $\mathcal{F}(x)$ is contained in the corresponding diagramme for $\mathcal{F}(y)$; whence there is a uniquely determined morphism

$$\mathcal{F}(x \leqslant y) : \mathcal{F}(y) \to \mathcal{F}(x)$$

that commutes with the arrows $\mathcal{F}(x) \to \mathcal{F}'(z)$ and $\mathcal{F}(y) \to \mathcal{F}'(z)$ where $z \in L'$ with $z \leqslant x$.

Convention In any context analogous to the one of the foregoing lemma, we write $\mathcal{F}(x) = \mathcal{F}'(x)$ in place of $\mathcal{F}(x) \cong \mathcal{F}'(x)$ whenever $x \in L'$.

The category of rings has finite inverse limits.

Definition 5 (ringed lattice) *A* ringed lattice *is a pair $X = (L, \mathcal{O})$ where L is a lattice and $\mathcal{O}$ is a sheaf of rings on L.*

Definition 6 (morphism of ringed lattices) *Let $X_1 = (L_1, \mathcal{O}_1)$, $X_2 = (L_2, \mathcal{O}_2)$ be ringed lattices. A* morphism of ringed lattices *$X_1 \to X_2$ is a pair (f, φ) consisting of a lattice homomorphism $f : L_1 \to L_2$ and a morphism $\varphi : \mathcal{O}_1 \to \mathcal{O}_2 \circ f$ of sheaves of rings on L_1: that is, a family of ring homomorphisms $\varphi(x) : \mathcal{O}_1(x) \to \mathcal{O}_2(f(x))$ with $x \in L_1$ which are compatible with the restriction mappings.*

The ringed lattices form a category, which also has finite inverse limits. Unlike the customary setting a morphism of ringed lattices (f, φ) has the same direction as its algebraic part φ, simply because its topological part f corresponds to the inverse image operator associated with a continuous mapping.

Let L be a lattice, $u \in L$, and L' the quotient of L modulo $u = 1$. Since the projection mapping $p_u : L \to L'$ satisfies

$$p_u(x) \leqslant p_u(y) \iff x \wedge u \leqslant y \wedge u$$

for all $x, y \in L$, this quotient L' can be identified with

$$\downarrow u = \{x \in L : x \leqslant u\}$$

and p_u with the mapping $x \mapsto x \wedge u$, where $\downarrow u$ has the lattice structure induced by that on L with the only exception that u stands for 1 in $\downarrow u$.

If $X = (L, \mathcal{O})$ is a ringed lattice, then so is

$$X|_u = (\downarrow u, \mathcal{O}|_{\downarrow u})$$

for every $u \in L$. If $(f, \varphi) : X_1 \to X_2$ is a morphism of ringed lattices, then so is

$$(f, \varphi)|_u = (f|_u, \varphi|_u) : X_1|_u \to X_2|_{f(u)}$$

for every $u \in L_1$ with $X_1 = (L_1, \mathcal{O}_1)$ and $X_2 = (L_2, \mathcal{O}_2)$, where

$$f|_{\downarrow u} : \downarrow u \to \downarrow f(u)\,, \quad \varphi|_u : \mathcal{O}_1|_{\downarrow u} \to \mathcal{O}_2|_{\downarrow f(u)} \circ f|_{\downarrow u}$$

are induced by f and φ, respectively.

Lemma 7 *Let $X = (L, \mathcal{O})$ be a ringed lattice. If $x_1, \ldots, x_n \in L$ with $1 = x_1 \vee \ldots \vee x_n$, then the canonical arrows $X \to X|_{x_i}$ induce an isomorphism*

$$X \cong \varprojlim \left\{ X|_{x_i} \to X|_{x_i \wedge x_j} : 1 \leqslant i, j \leqslant n \right\} .$$

In other words, for every family of morphisms of ringed lattices

$$Y \to X|_{x_i} \quad (1 \leqslant i \leqslant n)$$

that is compatible with the canonical arrows $X|_{x_i} \to X|_{x_i \wedge x_j}$, there is a uniquely determined morphism of ringed lattices $Y \to X$ that induces the given family.

Proof. Use first Lemma 5 of [15] for the lattices, and then that $\mathcal{O}$ is a sheaf on L. q.e.d.

In the opposite category of the category of ringed lattices the subobjects $X|_{x_i}$ hence form a cover of X.

2.3 Spectral Schemes

For every ring A we can now follow the usual definition of the structure sheaf as a sheaf of rings $\mathcal{O}_A$ on the Zariski lattice L_A. Using Lemma 4 we define $\mathcal{O}_A$ on the basis of L_A consisting of the $D(a)$ with $a \in A$. For every $a \in A$ let

$$S_a = \left\{c \in A : a \in \sqrt{(c)}\right\}$$

be the filter generated by a, and denote by

$$A_a = S_a^{-1} A \,,$$

the ring of fractions with denominators in S_a. There is the canonical ring homomorphism

$$\iota_a : A \to A_a \,, x \mapsto \frac{x}{1} \,,$$

which maps exactly the elements of S_a to the units of A_a; in particular, S_a contains all the units of A. Moreover, a ring homomorphism $\psi : A \to B$ factors through ι_a if and only if ψ maps S_a into the units of B, which is to say that $\psi(a)$ is a unit of B; in this case the factorisation is unique.

Note that S_a, A_a, and ι_a only depend on $D(a)$, because

$$S_a = \{c \in A : D(a) \leqslant D(c)\}$$

by (2). Furthermore, A_a is canonically isomorphic to $A\left[\frac{1}{a}\right]$, the ring of fractions whose denominators are the powers of a, and thus to $A[T]/(aT-1)$. Last but not least, the following three assertions are equivalent: a is a unit of A; S_a consists of the units of A; $\iota_a : A \to A_a$ is an isomorphism.

Now if $D(a) \leqslant D(b)$, then $S_b \subseteq S_a$; whence ι_a factors uniquely through ι_b: that is, there is a uniquely determined ring homomorphism

$$r_{a,b} : A_b \to A_a$$

with $r_{a,b} \circ \iota_b = \iota_a$. It is obvious that assigning A_a to $D(a)$ and $r_{a,b}$ to $D(a) \leqslant D(b)$ defines a presheaf of rings on the basis L'_A of L_A which consists of the $D(a)$ with $a \in A$. The computations from [34, pp. 125-6] show that this presheaf on L'_A actually is a sheaf on L'_A. By virtue of Lemma 4 we thus have:

Lemma 8 *There is a sheaf of rings $\mathcal{O}_A$ on L_A uniquely determined up to unique isomorphism such that*

$$\mathcal{O}_A(D(a)) = A_a$$

for every $a \in A$ and

$$\mathcal{O}_A(D(a) \leqslant D(b)) = r_{a,b}$$

whenever $D(a) \leqslant D(b)$.

Since $D(1) = 1$ and we have the canonical isomorpism

$$\iota_1 : A \xrightarrow{\cong} A_1 = \mathcal{O}_A(1) \ ,$$

an important particular case follows:

Corollary 9 *For every ring A we have $\mathcal{O}_A(1) \cong A$.*

For example, if A is a discrete domain, then

$$\mathcal{O}_A(D(a_1, \ldots, a_n)) = A\left[\frac{1}{a_1}\right] \cap \ldots \cap A\left[\frac{1}{a_n}\right]$$

within the quotient field of A.

Definition 10 (affine schemes) *For every ring A the* affine scheme $\operatorname{Spec} A$ *is the ringed lattice $(L_A, \mathcal{O}_A)$.*

Example 11 *If A is a ring and $a \in A$, then*

$$(\operatorname{Spec} A)\,|_{\downarrow D(a)} \cong \operatorname{Spec} A_a \, ;$$

in particular, this is an affine scheme.

To show that the construction of $\operatorname{Spec} A$ is functorial in A, let $\psi : A \to B$ be a ring homomorphism. Since there is exactly one lattice homomorphism $f_\psi : L_A \to L_B$ with

$$f_\psi(D(a)) = D(\psi(a)) \tag{6}$$

for every $a \in A$, the given ψ induces a family of ring homomorphisms

$$\mathcal{O}_A(D(a)) = A_a \to B_{\psi(a)} = \mathcal{O}_B(f_\psi(D(a)))$$

with $a \in A$, which are compatible with the restriction mappings. Again by Lemma 4 this family can be extended in precisely one way to a morphism of sheaves of rings

$$\varphi_\psi : \mathcal{O}_A \to \mathcal{O}_B \circ f_\psi \, .$$

In particular, $\varphi_\psi(1) : \mathcal{O}_A(1) \to \mathcal{O}_B(1)$ coincides with $\psi : A \to B$ modulo the canonical isomorphisms from Corollary 9.

Definition 12 (affine morphisms) *For every ring homomorphism $\psi : A \to B$ the* affine morphism $\operatorname{Spec}\psi : \operatorname{Spec} A \to \operatorname{Spec} B$ *is the morphism of ringed lattices (f_ψ, φ_ψ).*

One readily sees that Spec is a functor from the category of rings to the one of ringed lattices. By definition, Spec maps the category of rings surjectively—both on objects and morphisms—onto the subcategory of the category of ringed lattices which consists of the affine schemes and affine morphisms.

Definition 13 (spectral scheme) *A* spectral scheme *is a ringed lattice* $X = (L, \mathcal{O})$ *which is* locally affine*: that is, there are* $x_1, \ldots, x_n \in L$ *with* $1 = x_1 \vee \ldots \vee x_n$ *such that* $X|_{x_i} = (\downarrow x_i, \mathcal{O}|_{\downarrow x_i})$ *is isomorphic, as a ringed lattice, to* $\operatorname{Spec} \mathcal{O}(x_i)$ *for* $1 \leqslant i \leqslant n$. *Any finite sequence* $x_1, \ldots, x_n$ *of this kind is an* affine cover *of* X.

Lemma 14 *Every affine scheme is a spectral scheme.*

Proof. The singleton sequence 1 is an affine cover of $\operatorname{Spec} A$.. Q.E.D.

Lemma 15 *Let* $X = (L, \mathcal{O})$ *be a ringed lattice and* $x_1, \ldots, x_n \in L$. *If* $1 = x_1 \vee \ldots \vee x_n$ *and each* $X|_{x_i}$ *is a spectral scheme, then* X *is a spectral scheme.*

Proof. For $1 \leqslant i \leqslant n$, if $x_{i1}, \ldots, x_{ik_i}$ is an affine cover of $X|_{x_i}$, then

$$(\mathcal{O}|_{\downarrow x_i})\,|_{\downarrow x_{ij}} = \mathcal{O}|_{\downarrow x_{ij}}$$

for $1 \leqslant j \leqslant k_i$. Hence $x_{11}, \ldots, x_{1k_1}, \ldots, x_{n1}, \ldots, x_{nk_n}$ is an affine cover of X. Q.E.D.

2.4 Open Subschemes

Definition 16 (open subscheme of a spectral scheme) *If* $X = (L, \mathcal{O})$ *is a spectral scheme and* $u \in L$, *then* $X|_u = (\downarrow u, \mathcal{O}|_{\downarrow u})$ *is the* open subscheme *defined by* u.

The open subschemes of $\operatorname{Spec} A$ are of the form

$$(\operatorname{Spec} A)\,|_{D(a_1,\ldots,a_n)} = \left(\downarrow D(a_1, \ldots, a_n), \mathcal{O}_A|_{\downarrow D(a_1,\ldots,a_n)}\right) .$$

If $n = 1$, then this is an affine scheme, for $(\operatorname{Spec} A)\,|_{D(a)} \cong \operatorname{Spec} A_a$ (Example 11).

Lemma 17 *Every open subscheme of a spectral scheme is a spectral scheme.*

Proof. Let $X = (L, \mathcal{O})$ be a spectral scheme and $u \in L$, which is the top element of $\downarrow u$. As for $X|_u = (\downarrow u, \mathcal{O}|_{\downarrow u})$ being locally affine, we first consider the case in which $X = \operatorname{Spec} A$ is an affine scheme. In this case $u \in L_A$ is of the form $D(a_1) \vee \ldots \vee D(a_n)$ with $a_1, \ldots, a_n \in A$; whence $X|_u$ is a spectral scheme because $(\operatorname{Spec} A)\,|_{D(a_i)} \cong \operatorname{Spec} A_{a_i}$ (Example 11). In the general case, if $x_1, \ldots, x_n$ is an affine cover of X, then

$$u = (u \wedge x_1) \vee \ldots \vee (u \wedge x_n) .$$

Now each $u \wedge x_i$ defines the open subscheme

$$U_i = \left(\downarrow (u \wedge x_i)\,, (\mathcal{O}|_{\downarrow x_i})\,|_{\downarrow (u \wedge x_i)}\right)$$

of the affine scheme $(\downarrow x_i, \mathcal{O}|_{\downarrow x_i})$. By the first case, U_i is a spectral scheme for every i. In view of

$$(\mathcal{O}|_{\downarrow x_i})\,|_{\downarrow (u \wedge x_i)} = \mathcal{O}|_{\downarrow (u \wedge x_i)} = (\mathcal{O}|_{\downarrow u})\,|_{\downarrow (u \wedge x_i)} ,$$

every U_i is an open subscheme of $X|_u$, so that it suffices to invoke Lemma 15. Q.E.D.

2.5 Sheaves of Modules

Clearly, the category of abelian groups has finite inverse limits. First, let $X = (L, \mathcal{O})$ be a ringed lattice.

Definition 18 *A sheaf $\mathcal{M}$ on L with values in the category of abelian groups is a* sheaf of $\mathcal{O}$–modules*, or simply an $\mathcal{O}$–*module*, on X if $\mathcal{M}(x)$ is an $\mathcal{O}(x)$–module for every $x \in L$ such that the diagramme*

$$\begin{array}{ccc} \mathcal{O}(x) \times \mathcal{M}(x) & \to & \mathcal{M}(x) \\ \downarrow & & \downarrow \\ \mathcal{O}(y) \times \mathcal{M}(y) & \to & \mathcal{M}(y) \end{array}$$

is commutative whenever $x \geqslant y$. A sheaf of ideals on X *is an $\mathcal{O}$–submodule $\mathcal{I}$ of $\mathcal{O}$.*

Next, let A be a ring. Recall that if M is an A–module, then

$$M_a = S_a^{-1} M$$

is an A_a–module for every $a \in A$, where the filter S_a generated by a and the ring of fractions $A_a = S_a^{-1} A$ are as above. Note that M_a, just as S_a and A_a, depends only on $D(a)$.

For each A–module M we can now define, using Lemma 4, an $\mathcal{O}$–module $\widetilde{M}$ on the affine scheme $\operatorname{Spec} A = (L_A, \mathcal{O}_A)$ by setting

$$\widetilde{M}(D(a)) = M_a$$

for every $a \in A$, and by constructing the restriction mappings as for $\mathcal{O}_A$.

Finally, let $X = (L, \mathcal{O})$ be a spectral scheme.

Definition 19 *An $\mathcal{O}$–module $\mathcal{M}$ is* quasicoherent *if there is an affine cover $x_1, \ldots, x_n$ of X such that $\mathcal{M}|_{\downarrow x_i} \cong \widetilde{\mathcal{M}(x_i)}$.*

The quasicoherent $\mathcal{O}$–modules on X form an abelian category. Every quasicoherent $\mathcal{O}_A$–module on $\operatorname{Spec} A$ is isomorphic to some $\widetilde{M}$. In the following, we only consider quasicoherent sheaves of modules.

As a simple example, here is an explicit way of glueing locally defined submodules of a module without torsion over an integral domain.

Lemma 20 *Let A be an integral domain with (discrete) field of fraction K and V a K–vector space. We assume for given a covering $1 = D(a_1, \ldots, a_n)$ of L_A and a family of A_{a_i}–submodules M_i of V such that $M_i A_{a_i a_j} = M_j A_{a_i a_j}$ for all i, j. Then $M = M_1 \cap \ldots \cap M_n$ is the one and only A–submodule M of V such that $M A_{a_i} = M_i$ for each i.*

Proof. Set $M = M_1 \cap \ldots \cap M_n$. We prove $M A_{a_i} = M_i$ for each i. Since $M \subseteq M_i$ and M_i is an A_{a_i}–module, we have $M A_{a_i} \subseteq M_i$. Conversely, if m is in M_i, then we can find a number N such that for each j the element $a_j^N m$ is in M_j, because $M_i A_{a_i a_j} = M_j A_{a_i a_j}$.

Since $1 = D(a_1, \ldots, a_n)$, there are b_j in A such that $\Sigma b_j a_j^N = 1$; whence $m = \Sigma b_j a_j^N m$ is in M.

If M' is a A–submodule of V such that $M' A_{a_i} = M_i$, then $M' \subseteq M_i$ for each i and thus $M' \subseteq M$. Conversely, if m is in M, then we can find a number N such that $a_i^N m$ is in M' for each i. Since $1 = D(a_1, \ldots, a_n)$, there are b_j in A such that $\Sigma b_j a_j^N = 1$; whence $m = \Sigma b_j a_j^N m$ is in M'. Q.E.D.

2.6 Local Properties

Let A be a ring. Recall that $a_1, \ldots, a_n \in A$ are *comaximal* if $D(a_1, \ldots, a_n) = 1$ or, equivalently, if $1 \in (a_1, \ldots, a_n)$. We say that a property $E\,(A)$ of rings A (respectively, a property $E\,(M)$ of A–modules M) is *local* if the following two condition are satisfied:

1. $E\,(A)$ implies $E\,(A_a)$ (respectively, $E\,(M)$ implies $E\,(M_a)$) for all $a \in A$;
2. $E\,(A)$ (respectively, $E\,(M)$) holds whenever there are comaximal $a_1, \ldots, a_n \in A$ with $E\,(A_{a_i})$ (respectively, with $E\,(M_{a_i})$) for every i.

The following properties of rings are local: reduced; coherent; Noetherian; normal (every ideal is integrally closed); arithmetical (every finitely generated ideal is locally principal); pp-ring (every principal ideal is projective); pf-ring (every principal ideal is flat); Krull dimension $\leqslant k$ for a fixed integer $k \geqslant -1$ [8]; locally regular; semihereditary (every finitely generated ideal is projective); hereditary (semihereditary and Noetherian).

The following properties of rings fail to be local: local; integral; field; Bézout; Prüfer domain; Dedekind domain.

The following properties of modules are local: finitely generated; finitely presented; flat; finitely generated projective (or, equivalently, finitely presented flat); coherent; Noetherian.

Here is an explicit computation of a finite set of generators from generators given locally.

Lemma 21 *Let M be an A–module. If $a_1, \ldots, a_n \in A$ are comaximal and M_{a_i} is finitely generated as a A_{a_i}–module for each i, then M is finitely generated.*

Proof. For each i there is a finite subset S_i of M and a number N such that the elements s/a_i^N with s in S_i generate M_{a_i}. Let $m \in M$. For every i we can find a number N_i and m_i in the submodule of M generated by S_i such that $a_i^{N_i} m = m_i$. Since $1 = D(a_1, \ldots, a_n)$ it follows that m is a linear combination of the m_i. Hence $S_1 \cup \ldots \cup S_n$ generates M. Q.E.D.

Example 22 *It is* not *the case in general that if $M_{\mathfrak{p}}$ is finitely generated for each prime $\mathfrak{p}$ of A then M is finitely generated. For instance the $\mathbb{Z}$–submodule M of $\mathbb{Q}$ generated by all the $1/p$ where p is a prime number is not finitely generated, whereas $M_{\mathfrak{p}}$ clearly is a finitely generated $\mathbb{Z}_{\mathfrak{p}}$ module for each prime ideal $\mathfrak{p}$ of $\mathbb{Z}$.*

Definition 23 *Let E be a local property of rings. A spectral scheme $X = (L, \mathcal{O})$ has property E, for short $E(X)$, if there is an affine cover $x_1, \ldots, x_n$ of X with $E(\mathcal{O}(x_i))$ for every i.*

Note that $E(X)$ is equivalent to each of the following assertions:

1. there are $x_1, \ldots, x_n \in L$ with $1 = x_1 \vee \ldots \vee x_n$ such that $E(\mathcal{O}(x_i))$ for every i;
2. $E(\mathcal{O}(u))$ for every $u \in L$.

The $x_1, \ldots, x_n$ from assertion 1 above need not form an *affine* cover.

Definition 24 *Let E be a local property of modules, and $X = (L, \mathcal{O})$ a spectral scheme. A quasicoherent $\mathcal{O}$–module $\mathcal{M}$ has property E, for short $E(\mathcal{M})$, if there is an affine cover $x_1, \ldots, x_n$ of X with $E(\mathcal{M}(x_i))$ for every i.*

Let $X = (L, \mathcal{O})$ be a spectral scheme. If E is a local property of modules, and $\mathcal{M}$ a quasicoherent $\mathcal{O}$–module, then $E(\mathcal{M})$ is equivalent to each of the following assertions:

1. there are $x_1, \ldots, x_n \in L$ with $1 = x_1 \vee \ldots \vee x_n$ such that $E(\mathcal{M}(x_i))$ for every i;
2. $E(\mathcal{M}(u))$ for every $u \in L$.

Again, the $x_1, \ldots, x_n$ from assertion 1 above need not form an *affine* cover.

Definition 25 *An $\mathcal{O}$–module $\mathcal{M}$ is* coherent *if there is an affine cover $x_1, \ldots, x_n$ of X such that $\mathcal{M}|_{\downarrow x_i} \cong \widetilde{M_i}$ for suitable $\mathcal{O}(x_i)$–modules M_i that are finitely presented.*

For short, a coherent $\mathcal{O}$–module is one which is locally of finite presentation. This terminology, however, is in conflict with the notion of coherent module over a ring A: as an A-module whose finitely generated submodules are finitely presented. (In particular, a ring A is coherent if every finitely generated ideal is finitely presented: that is, if it is coherent as a module over itself.) As said above, the latter notion of coherence is a local property of $\mathcal{O}$–modules.

It is in order to point out that the notion of a coherent A-module plays an important role in constructive algebra [35, 32], where one often needs to add the hypothesis of coherence to achieve a constructive proof of a theorem about Noetherian modules. A prominent example is Noether's version of the Hilbert basis theorem; for recent developments in this area we refer to [38, 39, 44].

3 Further Examples

3.1 Projective Spaces

We now adapt [15] to the present setting. Let

$$A = \bigoplus_{d \geqslant 0} A^{(d)}$$

be a graded ring. We restrict ourselves to the common case (see, for instance, [20]) that A is generated as an $A^{(0)}$–algebra by finitely many $x_0, \ldots, x_n \in A^{(1)}$ with $n \geqslant 1$: that is,

$$A = A^{(0)}[x_0, \ldots, x_n] \, .$$

As usual, any $a \in A^{(d)}$ is called *homogeneous of degree d.*

Let P_A be the lattice generated by the symbols $D\,(a)$, with $a \in A$ homogeneous of degree > 0, which are subject to the relations

$$\begin{array}{l} D\,(x_0) \vee \ldots \vee D\,(x_n) = 1 \\ D(ab) = D(a) \wedge D(b) \\ D(0) = 0 \\ D(a+b) \leqslant D(a) \vee D(b) \end{array} \tag{7}$$

for all homogeneous $a, b \in A$ of degree > 0. Note that in the last condition of (7) the ring elements a and b have to have the same degree, to ensure that also $a+b$ is homogeneous. The elements of P_A are of the form

$$D(a_1, \ldots, a_n) = D(a_1) \vee \ldots \vee D(a_n)$$

with $a_1, \ldots, a_n \in A$ homogeneous of degree > 0.

As shown in [15], the lattice $P(A)$ is isomorphic to the quotient modulo $\sqrt{(x_0, \ldots, x_n)} = 1$ of the lattice formed by the radicals of finitely generated ideals whose generators are homogeneous elements of degree > 0. In particular, $D(a_1, \ldots, a_n)$ corresponds to $\sqrt{(a_1, \ldots, a_n)}$, and there also is a projective version of the formal Hilbert Nullstellensatz:

$$D(a_1, \ldots, a_n) \leqslant D(b_1, \ldots, b_m) \iff \sqrt{(a_1, \ldots, a_n)} \subseteq \sqrt{(b_1, \ldots, b_m)} \, .$$

Again, the $D\,(a)$ with $a \in A$ homogeneous of positive degree form a basis of P_A. Following the affine case, by Lemma 4 there is a sheaf of rings $\mathcal{O}^{(0)}$ on P_A such that

$$\mathcal{O}^{(0)}\,(D\,(a)) = (A_a)^{(0)}$$

and

$$\mathcal{O}^{(0)}\,(D\,(a) \leqslant D\,(b)) = (r_{a,b})^{(0)}$$

whenever $D\,(a) \leqslant D\,(b)$, where $a, b \in A$ are homogeneous of degree > 0.

Definition 26 (projective scheme) *For any graded ring A as above the* projective scheme Proj A *is the ringed lattice* $\left(P_A, \mathcal{O}^{(0)}\right)$.

Lemma 27 *Every projective scheme is a spectral scheme.*

PROOF. Let A be a graded ring as above. To prove that there is an affine covering, in view of $D(x_0) \vee \ldots \vee D(x_n) = 1$ it suffices to see that $(\text{Proj } A)|_{\downarrow D(x)} \cong \text{Spec}\,(A_x)^{(0)}$ for every $x \in A$ that is homogeneous of degree 1. This follows from [15, Proposition 3]. Q.E.D.

3.2 Spaces of Valuations

We first recall from [4] the essential concepts and results. Let K be a (discrete) field and R a ring with $R \subseteq K$. The lattice $\text{Val}_R(K)$ is generated by the symbols $V(s)$ with $s \in K$ which are subject to the relations

$$\begin{array}{l} V(r) = 1 \qquad (r \in R) \\ V(s) \wedge V(t) \leqslant V(s+t) \wedge V(st) \\ 1 = V(s) \vee V(1/s) \qquad (s \neq 0) \end{array}$$

where $s \in K$. The elements of $\text{Val}_R(K)$ are the finite joins of the $V(s_1) \wedge \ldots \wedge V(s_n)$.

There is a form of the formal Nullstellensatz also in this context:

Proposition 28 *We have $V(a_1) \wedge \ldots \wedge V(a_m) \leqslant V(s/t_1) \vee \ldots \vee V(s/t_n)$ if and only if s is integral over the ideal generated by $t_1, \ldots, t_n$ over the ring $R[a_1, \ldots, a_m]$.*

That s is integral over the ideal I generated by $t_1, \ldots, t_n$ means that we can find a relation $s^m + a_1 s^{m-1} + \ldots + a_m = 0$ with a_1 in I, …, a_m in I^m. In particular we have the following relation.

Corollary 29 *We have $V(a_1) \wedge \ldots \wedge V(a_m) \leqslant V(s)$ if and only if s is in the integral closure of the ring $R[a_1, \ldots, a_m]$ in K.*

The points of $\text{Val}_R(K)$ are the *valuation rings* V of K over R: that is, the subrings V of K with $R \subseteq V$ which satisfy

$$s \in V \vee 1/s \in V \quad (s \neq 0)\,.$$

In particular, the case $m = 0$ of Corollary 29 is a point-free version of the theorem that the intersection of all valuation rings of K over R is the integral closure of R in K.

If $R = k$ is a field, $s \in K$ transcendental over k, and K a finite algebraic extension of $k(s)$, then K is a field of algebraic functions of one variable over k. The valuation rings of K/k are the points of the abstract nonsingular curve over k with function field K.

In this case we define a sheaf of rings $\mathcal{O}$ on $\text{Val}_k(K)$ by

$$\mathcal{O}(x) = \{u \in K : x \leqslant V(u)\} \quad (x \in \text{Val}_k(K))\ .$$

An element $t \in K$ transcendental over k is called a *parameter*. The integral closure $E(t)$ of $k[t]$ in K is a Prüfer domain, and $\left(\downarrow V(t), \mathcal{O}|_{\downarrow V(t)}\right)$ is isomorphic to $\text{Spec}(E(t))$ [4]. Furthermore, we have $E(t) \subseteq E(t_1)$ whenever $V(t_1) \leqslant V(t)$.

Lemma 30 *The ringed lattice $X = (\mathrm{Val}_k(K), \mathcal{O})$ is a spectral scheme.*

Proof. See [4]. For every parameter t here is a two-element affine cover:

$$x_1 = V(t) \ , \ x_{-1} = V(t^{-1})$$
$$(\downarrow x_i, \mathcal{O}|_{\downarrow x_i}) \cong \mathrm{Spec}\, E(t^i)$$

where $E(t^i)$ is the integral closure of $k[t^i]$ in K for $i \in \{1, -1\}$. Q.E.D.

A (global) *divisor* on X is defined to be a $\mathcal{O}$–module $\mathcal{M}$ locally free of rank 1. We may assume that a divisor is given by an affine covering $x_1, \dots, x_n$ of $\mathrm{Val}_k(K)$ and non-zero elements $s_1, \dots, s_n$ of K with $\mathcal{M}(x_i) = s_i\mathcal{O}(x_i)$ such that

$$s_i\mathcal{O}(x_i \wedge x_j) = s_j\mathcal{O}(x_i \wedge x_j) \,.$$

The restriction of $\mathcal{M}$ to an open $V(t)$ is isomorphic to some $\widetilde{I_t}$, where I_t is a fractional ideal[2] of the domain $E(t)$. We can thus connect this definition of divisor with the one given by Edwards [19]: each divisor determines for every parameter t a fractional ideal I_t of the domain $E(t)$ such that if $V(t_1) \leqslant V(t)$, then $I_{t_1} = I_t E(t_1)$. Conversely, if I_t is a such a family, then the two fractional ideals $I_1 = I_s$ on $E(s)$ and $I_{-1} = I_{s^{-1}}$ on $E(s^{-1})$ are compatible, because $V(s) \wedge V(s^{-1}) = V(s + s^{-1})$ and

$$I_1\mathcal{O}(V(s) \wedge V(s^{-1})) = I_s E(s + s^{-1}) = I_{s+s^{-1}} = I_{s^{-1}} E(s + s^{-1}) = I_{-1}\mathcal{O}(V(s) \wedge V(s^{-1})) \,.$$

Moreover, any two compatible fractional ideals I_1, I_{-1} of that sort determine in a unique way a divisor. This is a consequence of the following result.

Lemma 31 *Let R be a Prüfer domain with fraction field K. Given two fractional ideals I_1 of $R[s]$ and I_{-1} of $R[s^{-1}]$ such that $I_1 R[s, s^{-1}] = I_{-1} R[s, s^{-1}]$ there exists one and only one fractional ideal I of R such that $IR[s] = I_1$ and $IR[s^{-1}] = I_{-1}$.*

Proof. The center map $c : L_R \to \mathrm{Val}_R(K)$, $D(a) \longmapsto V(1/a)$ is an isomorphism, and there exist u, v, w in R such that $c(D(u, w)) = V(s)$ and $c(D(1 - u, v)) = V(s^{-1})$ [4]. We then only need to glue two compatible fractional ideals defined locally on an affine scheme: that is, $I_1 R[1/u]$ on $D(u)$ and $I_{-1} R[1/(1 - u)]$ on $D(1 - u)$. We can take I to be generated by $u^N S_1 \cup (1 - u)^N S_{-1}$ where S_i generates I_i and N is a number such that $u^N S_i \subseteq I_{-i}$ (Lemma 20 and Lemma 21). Q.E.D.

4 Morphisms of Spectral Schemes

Definition 32 (morphism of spectral schemes) *Let $X_1 = (L_1, \mathcal{O}_1)$ and $X_2 = (L_2, \mathcal{O}_2)$ be spectral schemes. A* morphism of spectral schemes *is a morphism of ringed lattices $(f, \varphi) : X_1 \to X_2$ which is* locally affine*: that is, there are affine covers $x_1, \dots, x_n$ and $y_1, \dots, y_m$ of X_1 and X_2, respectively, which satisfy the following property:*

[2] We consider only *finitely generated* fractional ideals.

(*) *For every $j \leqslant m$ there is $i \leqslant n$ with $f(x_i) \geqslant y_j$ and such that the diagramme*

$$\begin{array}{ccccc} (\downarrow x_i, \mathcal{O}_1|_{\downarrow x_i}) & \xrightarrow[(f,\varphi)|_{x_i}]{} & (\downarrow f(x_i), \mathcal{O}_2|_{\downarrow f(x_i)}) & \xrightarrow[(p_{ij},\pi_{ij})]{} & (\downarrow y_j, \mathcal{O}_2|_{\downarrow y_j}) \\ \cong & & & & \cong \\ \operatorname{Spec}\mathcal{O}_1(x_i) & \xrightarrow[\operatorname{Spec}\varphi(x_i)]{} & \operatorname{Spec}\mathcal{O}_2(f(x_i)) & \xrightarrow[\operatorname{Spec}\mathcal{O}_2(f(x_i)\geqslant y_j)]{} & \operatorname{Spec}\mathcal{O}_2(y_j) \end{array} \tag{8}$$

is commutative with $p_{ij}(z) = z \wedge y_j$ and $\pi_{ij}(z) = \mathcal{O}_2(z \geqslant p(z))$ for every $z \in \downarrow f(x_i)$.

Note that there need not be a vertical arrow in the middle column of (8).

In the situation of Definition 32, for (f, φ) to be locally affine means that it locally is an affine morphism. In view of (6) this amounts to say that f locally is determined by φ: that is, locally and with the appropriate identifications we have

$$f \circ D = D \circ \varphi\,. \tag{9}$$

Lemma 33 *Let $X_1 = (L_1, \mathcal{O}_1)$ and $X_2 = (L_2, \mathcal{O}_2)$ be spectral schemes. If $(f, \varphi) : X_1 \to X_2$ is a spectral morphism, then so is $(f, \varphi)|_u : X_1|_u \to X_2|_{f(u)}$ for every $u \in L_1$.*

Proof. Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$ be as in Definition 32, and write $v = f(u)$. Note first that

$$u = (u \wedge x_1) \vee \ldots \vee (u \wedge x_n)\ , \quad v = (v \wedge y_1) \vee \ldots \vee (v \wedge y_m)\ ,$$

and recall that every open subscheme of a spectral scheme is a spectral scheme (Lemma 17). In particular, $X_1|_{u \wedge x_i}$ has an affine cover $x_{i1}, \ldots, x_{in_i}$ for every $i \leqslant n$, and $X_2|_{y_j \wedge f(x_{ik})}$ has an affine cover $y_{ikj1}, \ldots, y_{ikjm_{ikj}}$ for all $i \leqslant n$, $k \leqslant n(i)$, and $j \leqslant m$. Since v is the join of all the $y_j \wedge f(x_{ik})$, all the $y_{ikj\ell}$ form an affine cover of v with

$$\begin{array}{ccc} f(x_i) & \geqslant & y_j \\ \geqslant & & \geqslant \\ f(x_{ik}) & \geqslant & y_{ikj\ell} \end{array}$$

such that the diagramme (8) remains commutative if one replaces the x_i and the y_j with the x_{ik} and the $y_{ikj\ell}$, respectively. Q.E.D.

With Lemma 33 at hand it is straightforward to prove that a more flexible characterisation of a spectral morphism is equivalent to its very definition.

Corollary 34 *Let $X_1 = (L_1, \mathcal{O}_1)$, $X_2 = (L_2, \mathcal{O}_2)$ be spectral schemes. A morphism of ringed lattices $(f, \varphi) : X_1 \to X_2$ is locally affine precisely when for every affine cover $x_1, \ldots, x_n$ of X_1 there is an affine cover $y_1, \ldots, y_m$ of X_2 satisfying* (*).

The next corollary is a direct consequence of the foregoing.

Corollary 35 *The composition of two spectral morphisms is a spectral morphism.*

Definition 36 (open embedding) *Let $X = (L, \mathcal{O})$ be a spectral scheme and $u \in L$. The* open embedding $X \to X|_u$ *of the open subscheme defined by u consists of the lattice homomorphism*

$$L \to \downarrow u\,,\; x \mapsto x \wedge u$$

and the ring homomorphisms

$$\mathcal{O}\left(x \wedge u \leqslant x\right) : \mathcal{O}\left(x\right) \to \mathcal{O}\left(x \wedge u\right) \quad \left(x \in L\right)\,.$$

To prove the following is an easy exercise left to the reader.

Lemma 37 *Every open embedding is a spectral morphism.*

Complementing Lemma 14, the next lemma says that every affine morphism is a morphism of spectral schemes, and that every morphism of spectral schemes between affine schemes is an affine morphism. Recall from Corollary 9 that $A \cong \mathcal{O}_A(1)$ for every ring A.

Lemma 38 *If $\psi : A \to B$ is a ring homomorphism, then* $\operatorname{Spec}\psi : \operatorname{Spec} A \to \operatorname{Spec} B$ *is a morphism of spectral schemes. Conversely, if* $(f, \varphi) : \operatorname{Spec} A \to \operatorname{Spec} B$ *is a morphism of spectral schemes, then*

$$\psi : A \cong \mathcal{O}_A(1) \underset{\operatorname{Spec}\varphi(1)}{\longrightarrow} \mathcal{O}_B(1) \cong B\,.$$

is the one and only ring homomorphism from A to B such that $(f, \varphi) = \operatorname{Spec}\psi$.

PROOF. Let $X_1 = \operatorname{Spec} A$ and $X_2 = \operatorname{Spec} B$. The first part is plain: in Definition 32 take $n = 1$, $m = 1$, $x_1 = 1$, $y_1 = 1$. As for the second part, let (f, φ) a morphism of spectral schemes from X_1 to X_2, and $x_1, \ldots, x_n$ and $y_1, \ldots, y_m$ as in Definition 32. According to Lemma 7 there is exactly one morphism of ringed lattices from $\operatorname{Spec} A$ to $\operatorname{Spec} B$ that completes all the diagrammes

$$\begin{array}{ccccc} \operatorname{Spec} A & & \operatorname{Spec} B & \longrightarrow & (\downarrow y_j, \mathcal{O}_B|_{\downarrow y_j}) \\ \cong & & \cong & & \cong \\ \operatorname{Spec} \mathcal{O}_A(1) & \underset{\operatorname{Spec}\varphi(1)}{\longrightarrow} & \operatorname{Spec} \mathcal{O}_B(1) & \longrightarrow & \operatorname{Spec} \mathcal{O}_B(y_j)) \end{array} \tag{10}$$

with $j \leqslant m$. One readily sees that $\operatorname{Spec}\psi$ possesses this property; to show that it equals (f, φ) we only need to verify that the latter is of the same sort. To this end, let $j \leqslant m$ be given, and pick $i \leqslant n$ as in (*): that is, with $f(x_i) \geqslant y_j$ and such that (8) is commutative. Now (**??**) completed with (f, φ) factors through (8); whence the former is commutative for so is the latter. Q.E.D.

The spectral schemes and spectral morphisms form a category, of which affine schemes and affine morphisms form a full subcategory (Lemma 38). Since the two constructions given in Lemma 38 are clearly inverse to each other, we have the following:

Corollary 39 *The functor* Spec *induces an isomorphism*

$$\mathsf{Mor}(\operatorname{Spec} A, \operatorname{Spec} B) \cong \mathsf{Hom}(A, B)\,,$$

which is natural in rings A and B.

In other words, Spec is an equivalence from the category of rings to the aforementioned subcategory.

Example 40 (unit circle) *For every ring B there is a bijection*

$$\mathsf{Mor}\left(\operatorname{Spec} \mathbb{Z}\left[X, Y\right] / \left(X^2 + Y^2 - 1\right), \operatorname{Spec} B\right) \cong \left\{(x, y) \in B^2 : x^2 + y^2 = 1\right\}$$

More generally, $\operatorname{Spec} A$ has the expected universal property:

Proposition 41 *There is an isomorphism*

$$\mathsf{Mor}(\operatorname{Spec} A, X) \cong \mathsf{Hom}(A, \mathcal{O}\,(1))\,,$$

which is natural in rings A and in spectral schemes $X = (L, \mathcal{O})$.

Proof. Let A be a ring and $X = (L, \mathcal{O})$ a spectral scheme. Pick an affine cover $x_1, \ldots, x_n$ of X. Since $X|_{x_i} \cong \operatorname{Spec} \mathcal{O}(x_i)$, by Corollary 39 we have

$$\mathsf{Mor}(\operatorname{Spec} A, X|_{x_i}) \cong \mathsf{Hom}(A, \mathcal{O}(x_i))$$

for each i. Since $1 = x_1 \vee \ldots \vee x_n$, by Lemma 7 we have

$$X \cong \varprojlim \left\{X|_{x_i} \to X|_{x_i \wedge x_j} : 1 \leqslant i, j \leqslant n\right\}\,.$$

Putting all this together and using the fact that $\mathcal{O}$ is a sheaf, we obtain the required isomorphism, which by construction is natural both in A and in X. Q.E.D.

In other words, Spec is left adjoint to the functor which assigns to every spectral scheme $X = (L, \mathcal{O})$ the ring of global sections $\mathcal{O}(1)$. Note also that Corollary 39 is a special case of Proposition 41.

Example 42 (projective space) *Let $\mathbb{Z}\left[X_0, \ldots, X_n\right]$ be graded by degree. For every ring B there is a bijection between* $\mathsf{Mor}\,(\operatorname{Proj} \mathbb{Z}\left[X_0, \ldots, X_n\right], \operatorname{Spec} B)$ *and the B–modules of rank 1 which are direct summands of B^{n+1}.*

5 Classification of Spectral Schemes

By a *(Grothendieck) scheme* we understand a scheme as in customary algebraic geometry [23, 20], which is the framework of this last section. In particular, we are now reasoning classically: that is, with classical logic and the axiom of choice.

Recall that a *spectral space* is a topological space $\mathfrak{X}$ which is *sober* (that is, every nonempty irreducible closed subspace is the closure of a unique point, its generic point) and whose collection $\mathrm{K}(\mathfrak{X})$ of compact opens is a basis of the topology on $\mathfrak{X}$ and closed under finite intersection (that is, $\mathfrak{X} \in \mathrm{K}(\mathfrak{X})$, and if $U, V \in \mathrm{K}(\mathfrak{X})$, then $U \cap V \in \mathrm{K}(\mathfrak{X})$) .

In particular, every spectral space $\mathfrak{X}$ is compact and a Kolmogorov (T_0) space, for which $\mathrm{K}(\mathfrak{X})$ is a distributive lattice. The paradigmatic example of a spectral space is the affine spectrum $\mathfrak{Spec}(A)$ of a ring A, whose points are the prime ideals of A and for which $\mathrm{K}(\mathfrak{Spec}(A))$ is isomorphic to the Zariski lattice L_A described before.

Let $\mathfrak{X}_1$ and $\mathfrak{X}_2$ be spectral spaces. A *spectral mapping* is a continuous mapping $F : \mathfrak{X}_1 \to \mathfrak{X}_2$ for which $F^{-1}(V) \in \mathrm{K}(\mathfrak{X}_1)$ whenever $V \in \mathrm{K}(\mathfrak{X}_2)$. (This is already the case if F is *locally spectral*: that is, there is a finite covering $V_1, \ldots, V_n$ of $\mathfrak{X}_2$ such that the induced mappings $F^{-1}(V_i) \to V_i$ are spectral for each i.) In particular, every spectral mapping $F : \mathfrak{X}_1 \to \mathfrak{X}_2$ induces a lattice homomorphism $\mathrm{K}(F) : \mathrm{K}(\mathfrak{X}_1) \to \mathrm{K}(\mathfrak{X}_2)$.

The spectral spaces and mappings form a category, which is classically equivalent to the category of distributive lattices. One inverts the functor K as follows: if L is a distributive lattice, then the set $\mathfrak{Spec}\, L$ consisting of the prime filters of L is a spectral space, and if $f : L_1 \to L_2$ is lattice homomorphism, then $f^{-1} : \mathrm{Spec}\, L_2 \to \mathrm{Spec}\, L_1$ is a spectral mapping.

To extend this to schemes, all the necessary material was already present in [23]. In fact, three items of one section [23, Definition 6.1.1, Corollaire 6.1.10.iii, Proposition 6.1.12] suffice to verify the following lemma:

Lemma 43 *Let $F : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of Grothendieck schemes. If the topological space underlying $\mathfrak{X}$ is compact, and if $\mathrm{K}(\mathfrak{Y})$ is closed under binary intersection, then F is a spectral mapping.*

In particular, if $F : \mathfrak{X} \to \mathfrak{Y}$ is a morphism of Grothendieck schemes whose underlying topological spaces are spectral, then F is a spectral mapping. This is the only crucial point in the otherwise straightforward proof of the last proposition in this paper:

Proposition 44 (classical characterisation) *The category of spectral schemes and spectral morphisms is equivalent to the full subcategory of the category of Grothendieck schemes whose objects have spectral spaces as underlying topological spaces.*

The following combination of [23, Proposition 2.1.5, Corollaire 6.1.13] and [23, 6.1] hinted us first at the foregoing proposition.

Remark 45 *Every scheme is a sober space, and every Noetherian scheme is a spectral space. If $\mathfrak{X}$ is a Noetherian scheme, then every morphism of schemes $f : \mathfrak{X} \to \mathfrak{Y}$ is a spectral mapping.*

As a consequence, the spectral schemes conceived in this paper, based on distributive lattices rather than topological spaces, are sufficient for dealing with Noetherian schemes.

6 On Locality

Since a spectral scheme locally is an affine scheme, its structure sheaf $\mathcal{O}$ is a sheaf of local rings also in the point-free meaning of this notion [24]: if $s, t \in \mathcal{O}(y)$ with $s + t$ invertible, then there are $y_1, \ldots, y_m \in L$ with $y = y_1 \vee \ldots \vee y_m$ such that for every j either s or t is invertible in $\mathcal{O}(y_j)$. In particular, when $\mathcal{O}$ is viewed from the customary perspective [23, 20], then it gives rise to a sheaf of local rings: that is, the stalks of $\mathcal{O}$ are all local rings.

Moreover, in view of (**??**) the spectral morphisms automatically satisfy an appropriate counterpart of the locality condition required in from the morphisms of formal geometries [43]. Formal geometries are the point-free version of locally ringed spaces that has been developed on the basis of formal topology [40, 41]. From the customary perspective, in particular, spectral morphisms are morphisms of locally ringed spaces: that is, morphisms which on the local rings induce local homomorphisms.

All this notwithstanding we have decided not to introduce the notion of a locally ringed lattice, nor the one of a morphism of locally ringed lattices. There is a natural way to do so, by which one arrives at a category equivalent to a full subcategory of the category of formal geometries: the one whose objects are based on finitary formal topologies, the counterparts of distributive lattices [36, 37, 40]. It has turned out, however, that in the context of spectral schemes and spectral morphisms, which both are "local" by definition, up to a certain point one can get by on without any talk of locally ringed lattices and their morphisms, concepts which seem relatively involved if compared with the ones given in this paper.

Locally ringed lattices and their morphisms still need to be studied, including a thorough analysis of their connections with the category of formal geometries. It also remains to be seen whether in the universal property which distinguishes the affine schemes among the spectral schemes (Proposition 41 above) the notion of a morphism of spectral schemes can be widened from the the one of a spectral morphism to the more general one of a morphism of locally ringed lattices. Only thus we would achieve a result fully analogous to the universal property within formal geometries [43], and hence to the well known one from the customary setting.

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